Exam in Mathematics for Multimedia Applications

First Year at the Technical Faculty of IT and Design

13 August 2019, 9:00-13:00

This exam set consists of 8 pages with 12 problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes, photocopies etc. It is **not allowed** to use **any electronic devices** such as pocket calculators, mobile phones, or computers.

The exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains "multiple choice" problems. **The answers of Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number below. Also write name and student number on each page of your solutions of the essay problems and number these pages. Indicate the total number of extra sheets on the first page.

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

Answers

Part I (Essay problems)

Problem 1 (9 points)

(a) (5 points). Prove that the following trigonometric identity holds:

$$\frac{\sin(x)+\cos(x)}{\cos(x)-\sin(x)} = \tan(x+\tfrac{\pi}{4}).$$

(b) (4 points). Describe all solutions of the equation

$$\frac{\cos(x)+\sin(x)}{\cos(x)-\sin(x)} = \sqrt{3}.$$

SOLUTION:

(a) Employing the addition theorems for the trigonometric functions in the second step, we find

$$\begin{aligned}\tan(x+\tfrac{\pi}{4}) &= \frac{\sin(x+\frac{\pi}{4})}{\cos(x+\frac{\pi}{4})}\\ &= \frac{\sin(x)\cos(\frac{\pi}{4})+\cos(x)\sin(\frac{\pi}{4})}{\cos(x)\cos(\frac{\pi}{4})-\sin(x)\sin(\frac{\pi}{4})}.\end{aligned}$$

Now we recall that

$$\sin(\tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) = \frac{1}{\sqrt{2}},$$

which together with the above computation permits to get

$$\begin{aligned}\tan(x+\tfrac{\pi}{4}) &= \frac{\sin(x)\cdot\frac{1}{\sqrt{2}}+\cos(x)\cdot\frac{1}{\sqrt{2}}}{\cos(x)\cdot\frac{1}{\sqrt{2}}-\sin(x)\cdot\frac{1}{\sqrt{2}}}\\ &= \frac{\frac{1}{\sqrt{2}}\cdot(\sin(x)+\cos(x))}{\frac{1}{\sqrt{2}}\cdot(\cos(x)-\sin(x))}\\ &= \frac{\cos(x)+\sin(x)}{\cos(x)-\sin(x)}.\end{aligned}$$

(b) In view of Part (a) the given equation is equivalent to

$$\tan(x+\tfrac{\pi}{4}) = \sqrt{3}.$$

Now we know that $\tan(y) = \sqrt{3}$, if and only if $y = \frac{\pi}{3}+n\pi$ for some $n\in\mathbb{Z}$. Setting $y = x+\frac{\pi}{4}$, that is, $x = y-\frac{\pi}{4}$, we see that x solves the given equation, if and only if

$$x = \frac{\pi}{3}+n\pi-\frac{\pi}{4} = \frac{(4-3)\pi}{12}+n\pi = \frac{\pi}{12}+n\pi,$$

for some $n\in\mathbb{Z}$. $\square$

Problem 2 (19 points)

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 1, \\ x_1 + 2x_2 + 3x_3 &= 1, \\ 2x_1 + x_2 + 4x_3 &= 1. \end{aligned} \tag{1}$$

(a) (2 points). Write down the augmented matrix corresponding to the system of linear equations (1).

(b) (10 points). Compute the reduced row echelon form of the augmented matrix corresponding to the system (1).

(c) (4 points). Find all solutions (if any) to the system (1).

(d) (3 points). Find all solutions (if any) to the system

$$\begin{aligned} x_1 + 2x_3 + x_4 &= 1, \\ x_1 + 2x_2 + 3x_3 &= 1, \\ 2x_1 + x_2 + 4x_3 &= 1, \\ x_1 - 5x_4 &= 2019. \end{aligned}$$

SOLUTION

(a) The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 3 & 0 & 1 \\ 2 & 1 & 4 & 0 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 3 & 0 & 1 \\ 2 & 1 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 = r_2^{\text{new}}} \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & -1 & 0 \\ 2 & 1 & 4 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - 2r_1 = r_3^{\text{new}}}$$
$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & -1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 2 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{r_3 - 2r_2 = r_3^{\text{new}}}$$
$$\begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix} \xrightarrow{r_1 - 2r_3 = r_1^{\text{new}}} \begin{bmatrix} 1 & 0 & 0 & -5 & -3 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

(c) The set of all solutions is given by

$$\left\{ \begin{bmatrix} -3 + 5x_4 \\ -1 + 2x_4 \\ 2 - 3x_4 \\ x_4 \end{bmatrix} \middle| \; x_4 \in \mathbb{R} \right\}.$$

(Alternatively, one could write the answer as follows:

$$x_1 = -3 + 5x_4, \quad x_2 = -1 + 2x_4, \quad x_3 = 2 - 3x_4, \quad x_4 \text{ free.})$$

(d) Since the first three lines of the given system of four equations are identical to the system solved in Parts (a) through (c), any solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

of the given system of four equations must be contained in the set of solutions found in Part (c) and, in particular, it must satisfy $x_1 = -3 + 5x_4$. But then

$$x_1 - 5x_4 = -3 + 5x_4 - 5x_4 = -3 \neq 2019.$$

We conclude that the given system of four equations cannot have any solution.

(One can also solve Part (d) by writing down a new, 4×4 augmented matrix and computing its reduced row echelon form, from which it can be read off that no solution exists. The above argument is, however, somewhat shorter.) $\square$

Part II (Multiple choice problems)

Problem 3 (6 points)

Let a be a real constant. Mark the correct expression for the limit

$$\lim_{h \to 0} \frac{2\sin((a+h)^3) - 2\sin(a^3)}{h}.$$

☐ $6\sin(a^3)$	☑ $6a^2\cos(a^3)$
☐ $2\sin(a^3)$	☐ $2\cos(a^3)$
☐ $6a^2\sin(a^3)$	☐ $2\tan(a^3)$

Problem 4 (6 points)

A function is given by

$$f(x) = 3x(\ln(x) - 1).$$

Mark the correct expression for its derivative $f'(x)$.

☑ $\ln(x^3)$	☐ $\frac{3-\ln(x^3)}{x^2}$
☐ $\frac{3}{x}$	☐ $\frac{1-3\ln(x)}{x^2}$
☐ $\frac{3\frac{1}{x}-3}{x^2}$	☐ $3e^x$

Problem 5 (6 points)

A function is defined by

$$g(x) = \exp(6x^2 - 3x + 19).$$

The graph of the function has a horizontal tangent at a point. What is the x-coordinate of that point?

- ☑ $1/4$
- ☐ 3
- ☐ 1
- ☐ 4
- ☐ -2
- ☐ 0

Problem 6 (6 points)

A particle is moving along a horizontal x-axis. Its velocity as a function of time is given by $v(t) = 6te^t + 2t$. At time $t = 0$, the particle is located at $x = -6$. What is the position function for the particle?

- ☐ $x(t) = e^{6t} + t^2 - 7$
- ☐ $x(t) = 6te^t + 2t + 4$
- ☐ $x(t) = 6te^{6t} + 2t - 6$
- ☐ $x(t) = 6e^t + 2t - 2$
- ☐ $x(t) = 6te^t + t^2 + 4$
- ☑ $x(t) = 6te^t + t^2 - 6e^t$

Problem 7 (4 points)

The integral

$$\int_{-100}^{100} x^3 dx$$

is equal to

- ☑ 0
- ☐ 2000000
- ☐ -10000000
- ☐ 25000000
- ☐ 10000000
- ☐ 50000000

Problem 8 (5 points)

Mark the value of the sum

$$\sum_{i=0}^{4} (i^2 - 5).$$

- ☐ 30
- ☑ 5
- ☐ 0
- ☐ 55
- ☐ 10
- ☐ -30

Problem 9 (6 points)

Mark the value of the sum

$$\sum_{i=1}^{10} 3(i^3 - i).$$

- [ ] 8200
- [ ] 5200
- [x] 8910
- [ ] 9020
- [ ] 8120
- [ ] 6400

Problem 10 (11 points)

Four points in 3D-space are given by

$$P = (-1, 0, 0), \quad Q = (0, -1, 0), \quad R = (3, 2, 2), \quad S = (2, 0, 1).$$

In consequence, we have the following two vectors

$$\overrightarrow{PQ} = (1, -1, 0), \quad \overrightarrow{RS} = (-1, -2, -1).$$

Mark the correct answers below.

(a) (2 points). The coordinates of the vector $\overrightarrow{PR}$ are

- [ ] $(1, -1, 2)$
- [ ] $(4, 2, 4)$
- [x] $(4, 2, 2)$
- [ ] $(3, -1, -4)$

(b) (3 points). The line ℓ_1 through P and Q has parametric equation

- [x] $(x, y, z) = (-1, 0, 0) + t(1, -1, 0)$
- [ ] $(x, y, z) = (-3, 1, 1) + t(1, -1, 0)$
- [ ] $(x, y, z) = (-2, 0, 1) + t(-4, -1, 1)$
- [ ] $(x, y, z) = (-3, 1, 1) + t(-2, 0, 1)$

(c) (3 points). One finds that the line ℓ_2 through R and S has parametric equation

$$(x, y, z) = (3, 2, 2) + s(1, 2, 1)$$

The two lines ℓ_1 and ℓ_2 intersect at the point $T = (1, -2, 0)$. Which value of the parameter s corresponds to this intersection point?

- [x] $s = -2$
- [ ] $s = 2$
- [ ] $s = -3$
- [ ] $s = 3$

(d) (3 points). What is the cross product $\overrightarrow{PQ} \times \overrightarrow{RS}$?

- [x] $(1, 1, -3)$
- [ ] $(2, 2, -3)$
- [ ] $(-2, -3, -2)$
- [ ] $(3, 0, 1)$

Problem 11 (10 points)

(a) (5 points). Consider the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1, \\ 6x_1 + 9x_2 + 3x_3 &= 1, \\ x_3 - x_4 &= 0. \end{aligned}$$

Mark the correct statement on this system.

- [x] The system has no solution.
- [ ] The system has exactly one solution.
- [ ] The system has exactly four solutions.
- [ ] The system has infinitely many solutions.

(b) (5 points). Consider the system of linear equations

$$\begin{aligned} x_1 + ax_2 + ax_3 &= 1, \\ x_2 + ax_3 &= 2, \\ x_3 &= 3, \end{aligned}$$

where a is some real number. Mark the correct statement:

- [ ] The system has exactly one solution when $a = 0$, but for all other values of a it does not have any solution.
- [ ] The system has exactly one solution when $a = 1$, but for all other values of a it does not have any solution.
- [ ] The system has infinitely many solutions when $a = 1$.
- [x] The system has exactly one solution for each given value of a.
- [ ] There doesn't exist any value of a for which the system has a solution.

Problem 12 (12 points)

Decide whether the following statements are true or false.

(a) (2 points). For all $n \times n$-matrices A and B and real numbers c, it holds

$$c(AB)^T = B^T(cA)^T.$$

☑ True ☐ False

(b) (2 points). For all invertible $n \times n$-matrices A and B and real numbers c, it holds

$$c(AB)^{-1} = B^{-1}(cA)^{-1}.$$

☐ True ☑ False

(c) (2 points). Let R_ϕ and R_θ be 2×2-rotation matrices corresponding to angles ϕ and θ, respectively. Then it holds $R_\phi R_\theta^{-1} = R_{\phi-\theta}$.

☑ True ☐ False

(d) (2 points). Let R_ϕ be a 2×2-rotation matrix corresponding to the angle ϕ. Then it holds $R_\phi^{-1} = R_\phi^T$.

☑ True ☐ False

(e) (2 points). The following matrix represents a rotation about the angle $3\pi/2$ in the counter-clockwise direction:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

☐ True ☑ False

(f) (2 points). Let a, b, c, d be real numbers with $ad - bc = 1$. Then the inverse of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

☑ True ☐ False