Exam in Mathematics for Multimedia Applications

First Year at the Technical Faculty of IT and Design

3 June 2019, 9:00-13:00

This exam set consists of 8 pages with 12 problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use books, notes, photocopies etc. It is **not allowed** to use **any electronic devices** such as pocket calculators, mobile phones, or computers.

The exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains "multiple choice" problems. **The answers of Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number below. Also write name and student number on each page of your solutions of the essay problems and number these pages. Indicate the total number of extra sheets on the first page.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Answers

Part I (Essay problems)

Problem 1 (9 points)

(a) (5 points). Prove that the following trigonometric identity holds:

$$1 - \frac{\sin(x)}{\sin(\frac{3\pi}{2} - x)} = \tan(x) + 1.$$

(b) (4 points). Describe all solutions of the equation

$$1 - \frac{\sin(x)}{\sin(\frac{3\pi}{2} - x)} = 2.$$

SOLUTION:

(a) Employing an addition theorem for the trigonometric functions we find

$$\begin{aligned}\sin\left(\frac{3\pi}{2} - x\right) &= \sin\left(\frac{3\pi}{2}\right)\cos(x) - \cos\left(\frac{3\pi}{2}\right)\sin(x)\\ &= (-1)\cdot\cos(x) - 0\cdot\sin(x) = -\cos(x),\end{aligned}$$

for all $x \in \mathbb{R}$. Since $\tan(x) = \sin(x)/\cos(x)$, this implies

$$1 - \frac{\sin(x)}{\sin(\frac{3\pi}{2} - x)} = 1 - \frac{\sin(x)}{-\cos(x)} = 1 + \frac{\sin(x)}{\cos(x)} = 1 + \tan(x),$$

for all $x \in \mathbb{R}$ such that $x \notin \{\frac{\pi}{2} + n\pi | n \in \mathbb{Z}\}$.

(b) In view of Part (a) the given equation is equivalent to

$$\tan(x) + 1 = 2, \quad \text{that is, to} \quad \tan(x) = 1.$$

Therefore, x solves the given equation, if and only if $x = \frac{\pi}{4} + n\pi$ for some $n \in \mathbb{Z}$.

□

Problem 2 (17 points)

Consider the following invertible 3×3-matrix,

$$A := \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}.$$

(a) (10 points). Find A^{-1}.

(b) (3 points). Find all solutions (if any) of the following system of linear equations,

$$
\begin{aligned}
x_1 + 2x_2 &= 1, \\
x_1 + 3x_2 + 5x_3 &= 1, \\
2x_1 + 4x_2 + x_3 &= 1.
\end{aligned}
$$

(c) (4 points). Find all solutions (if any) of the following system of linear equations,

$$
\begin{aligned}
x_1 + 2x_2 &= 1, \\
x_1 + 3x_2 + 5x_3 &= 1, \\
2x_1 + 4x_2 + x_3 &= 1, \\
x_2 + x_3 &= 2019.
\end{aligned}
$$

SOLUTION

(a)

$$
\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 = r_2^{\text{new}}} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 - 2r_1 = r_3^{\text{new}}}
$$

$$
\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 - 5r_3 = r_2^{\text{new}}} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 9 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 2r_2 = r_1^{\text{new}}}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & -17 & -2 & 10 \\ 0 & 1 & 0 & 9 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix}
$$

This shows that

$$
A^{-1} = \begin{bmatrix} -17 & -2 & 10 \\ 9 & 1 & -5 \\ -2 & 0 & 1 \end{bmatrix}.
$$

(b) The given system is equivalent to

$$
Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{1}
$$

Since A is invertible, we further know that (1) is equivalent to

$$
x = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -17 & -2 & 10 \\ 9 & 1 & -5 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ -1 \end{bmatrix}.
$$

(Of course, this part of the problem can also be solved by applying the Gauss algorithm to the augmented matrix, but we can save work by applying the result of (a).)

(c) The first three lines of the given system are identical to the system solved in Part (b). Therefore, the only possible candidate for a solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ -1 \end{bmatrix}.$$

The last two components of x do, however, not satisfy the fourth equation,

$$5 + (-1) \neq 2019.$$

Hence, the given system of four equations does not have any solution.

(Of course, this part of the problem can also be solved by applying the Gauss algorithm to the augmented matrix, but we can save work by applying the result of (b).)

$\square$

Part II (Multiple choice problems)

Problem 3 (6 points)

Let a be a real constant. Mark the correct expression for the limit

$$\lim_{h \to 0} \frac{e^{2(a+h)} - e^{2a}}{h}.$$

- ☐ e^{2a}
- ☐ e^2
- ☐ $2e^a$
- ☑ $2e^{2a}$
- ☐ $2e$
- ☐ ae^{2a}

Problem 4 (6 points)

A function is given by

$$f(x) = \frac{3\ln(x)}{x}.$$

Mark the correct expression for its derivative $f'(x)$.

- ☐ $\frac{3}{x^2}$
- ☑ $\frac{3-\ln(x^3)}{x^2}$
- ☐ $\frac{3}{x}$
- ☐ $\frac{1-3\ln(x)}{x^2}$
- ☐ $\frac{3\frac{1}{x}-3}{x^2}$
- ☐ $3e^x$

Problem 5 (6 points)

A function is defined by

$$g(x) = \ln(x^2 + 6x + 10).$$

The graph of the function has a horizontal tangent at a point. What is the x-coordinate of that point?

- [x] -3
- [ ] 3
- [ ] 1
- [ ] 5
- [ ] -2
- [ ] 0

Problem 6 (6 points)

A particle is moving along a horizontal x-axis. Its velocity as a function of time is given by $v(t) = 6e^t + 2$. At time $t = 0$, the particle is located at $x = 4$. What is the position function for the particle?

- [ ] $x(t) = e^{6t} + 2t + 3$
- [ ] $x(t) = 6te^t + 2t + 4$
- [ ] $x(t) = 6te^{6t} + 2t - 4$
- [x] $x(t) = 6e^t + 2t - 2$
- [ ] $x(t) = 6te^t + t^2 + 4$
- [ ] $x(t) = 6e^{6t} + 2t^2 - 2$

Problem 7 (4 points)

The integral

$$\int_0^{\pi/2} (\sin(x) + \sin(-x))dx$$

is equal to

- [x] 0
- [ ] 2
- [ ] -1
- [ ] π
- [ ] 1
- [ ] 2π

Problem 8 (5 points)

Mark the value of the sum

$$\sum_{i=2}^{6}(2i^2+5i).$$

- ☐ 270
- ☐ 285
- ☑ 280
- ☐ 256
- ☐ 264
- ☐ 274

Problem 9 (6 points)

Mark the value of the sum

$$\sum_{i=1}^{10}(3i^3-i).$$

- ☐ 8200
- ☐ 5200
- ☐ 8410
- ☑ 9020
- ☐ 8120
- ☐ 6400

Problem 10 (11 points)

Four points in 3D-space are given by

$$P=(-3,1,1),\quad Q=(-2,0,1),\quad R=(1,3,5),\quad S=(0,1,3).$$

In consequence, we have the following two vectors

$$\overrightarrow{PQ}=(1,-1,0),\quad \overrightarrow{RS}=(-1,-2,-2).$$

Mark the correct answers below.

(a) (2 points). The coordinates of the vector $\overrightarrow{PR}$ are

- ☐ $(1,-1,2)$
- ☑ $(4,2,4)$
- ☐ $(3,-1,1)$
- ☐ $(3,-1,-4)$

(b) (3 points). The line ℓ_1 through P and Q has parametric equation

- ☐ $(x,y,z)=(0,1,3)+t(1,-1,2)$
- ☑ $(x,y,z)=(-3,1,1)+t(1,-1,0)$
- ☐ $(x,y,z)=(-2,0,1)+t(-4,-1,1)$
- ☐ $(x,y,z)=(-3,1,1)+t(-2,0,1)$

(c) (3 points). One finds that the line ℓ_2 through R and S has parametric equation

$$(x, y, z) = (1, 3, 5) + s(-1, -2, -2)$$

The two lines ℓ_1 and ℓ_2 intersect at the point $T = (-1, -1, 1)$. Which value of the parameter s corresponds to this intersection point?

- ☐ $s = -2$
- ☑ $s = 2$
- ☐ $s = -3$
- ☐ $s = 3$

(d) (3 points). What is the cross product $\overrightarrow{PQ} \times \overrightarrow{RS}$?

- ☐ $(1, 1, 3)$
- ☑ $(2, 2, -3)$
- ☐ $(-2, -3, -2)$
- ☐ $(3, 0, 1)$

Problem 11 (10 points)

(a) (5 points). Consider the system of linear equations

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1, \\ 4x_1 + 2x_2 + 7x_3 &= 1. \end{aligned}$$

Mark the correct statement on this system.

- ☐ The system has no solution.
- ☐ The system has exactly one solution.
- ☐ The system has exactly two solutions.
- ☑ The system has infinitely many solutions.

(b) (5 points). Consider the system of linear equations

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1, \\ 4x_1 + 2x_2 + 7x_3 &= 1, \\ 4x_1 + 2x_2 + 8x_3 &= a, \end{aligned}$$

where a is some real number. For which value of a does this system have at least one solution?

- ☐ 1
- ☑ 0
- ☐ -2
- ☐ 8

- ☐ There doesn't exist any value of a for which the system has a solution.

Problem 12 (14 points)

We consider the 2×2-matrix

$$A := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Mark the correct answers below.

(a) (4 points). The matrix A represents

- ☐ A reflection at the line $x = y$ in the x-y-plane.
- ☐ A reflection at the line $x = -y$ in the x-y-plane.
- ☐ A rotation in the x-y-plane about the angle $\pi/4$ in the clockwise direction.
- ☑ A rotation in the x-y-plane about the angle $\pi/4$ in the counter-clockwise direction.
- ☐ A rotation in the x-y-plane about the angle $\pi/3$ in the clockwise direction.
- ☐ A rotation in the x-y-plane about the angle $\pi/3$ in the counter-clockwise direction.

(b) (5 points). $(A^T)^{-1}$ is equal to

- ☐ $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
- ☐ $\sqrt{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- ☑ $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- ☐ $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$
- ☐ A^T is not invertible, so that $(A^T)^{-1}$ simply doesn't exist.

(c) (5 points). For which integer n do we have $A^n = I_2$?

☐ 3 ☐ -4 ☐ 6 ☑ 8

☐ There doesn't exist any integer n for which $A^n = I_2$.