Exam in Mathematics for Multimedia Applications

First Year at the Technical Faculty of IT and Design

29 May 2017

This exam set consists of 8 pages with 13 problems. A number of points is indicated for each question. The total number of points equals 100.

It is allowed to use books, notes, photocopies etc. It is **not allowed** to use **any electronic devices** such as pocket calculators, mobile phones or computers.

The exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains "multiple choice" problems. **The answers of Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number below. Also write name and student number on each page of your solutions of the essay problems and number these pages. Indicate the total number of extra sheets on the first page.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Answers

Part I (Essay-problems)

Problem 1 (8 points)

(a) (2 points). Prove that the following identity holds:

$$(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = \cos^4 x - \sin^4 x.$$

(b) (3 points). Prove the trigonometric identity

$$\cos(2x) = \cos^4 x - \sin^4 x.$$

(c) (3 points). Describe all solutions of the equation

$$\cos^4 x = \sin^4 x.$$

(a) Follows from the relation $(a+b)(a-b) = a^2 - b^2$.

(b) Follows by (a) since $\cos^2 x + \sin^2 x = 1$ and $\cos(2x) = \cos^2 x - \sin^2 x$.

(c) $x = \frac{\pi}{4} + \frac{\pi}{2}p, \quad p \in \mathbb{Z}$.

Problem 2 (12 points)

A system of linear equations is given by

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 7 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ x_1 + 3x_2 - x_3 &= 8. \end{aligned}$$

(a) (3 points). Find the augmented matrix of the system.

(b) (5 points). Find the reduced row echelon form of the augmented matrix.

(c) (4 points). Write down the general solution of the system.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 7 \\ 2 & 1 & 3 & 1 \\ 1 & 3 & -1 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} &x_1 = -1 - 2x_3 \\ &x_2 = 3 + x_3 \\ &x_3 \text{ free} \end{aligned}$$

Part II (Multiple-choice problems)

Problem 3 (5 points)

A particle is moving along a horizontal axis. Its position as a function of time t is given by

$$x(t) = t^2 - 6t + 4.$$

What is the position of the particle when its velocity is zero?

- ☐ 4
- ☑ -5
- ☐ -1
- ☐ 2
- ☐ 0
- ☐ 11

Problem 4 (4 points)

A function is given by

$$f(x) = \cos(\frac{1}{x}).$$

Mark the correct expression for its derivative $f'(x)$.

- ☐ $-\sin(\ln(x))$
- ☐ $-\sin(\frac{1}{x})$
- ☑ $\frac{1}{x^2}\sin(\frac{1}{x})$
- ☐ $\sin(\frac{1}{x})$
- ☐ $\sin(\frac{1}{x^2})$
- ☐ $-\ln(x)\sin(\frac{1}{x})$

Problem 5 (6 points)

A function is defined by

$$g(x) = \ln(2x^2 + x + 2).$$

The graph of the function has a tangent at the point $(1, g(1))$. What is the slope of that tangent?

- ☐ 2
- ☐ 3
- ☐ 0
- ☑ 1
- ☐ -1
- ☐ 5

Problem 6 (8 points)

Evaluate the integrals below and mark the correct results.

(a) (4 points). The integral

$$\int_0^{\pi} (2x + \cos x) dx$$

is equal to

- [x] π^2
- [ ] 2π
- [ ] $\pi^2 + 2$
- [ ] 2
- [ ] 0
- [ ] $2\pi - 1$

(b) (4 points). The integral

$$\int_1^3 \frac{1}{x^2} dx$$

is equal to

- [ ] $-\frac{8}{9}$
- [ ] 1
- [ ] $\frac{1}{2}$
- [ ] $\frac{25}{27}$
- [x] $\frac{2}{3}$
- [ ] $\frac{10}{9}$

Problem 7 (3 points)

Mark the value of the sum

$$\sum_{i=1}^{6} i(i-3).$$

- [ ] 20
- [ ] 9
- [ ] 42
- [ ] 34
- [ ] 25
- [x] 28

Problem 8 (6 points)

Mark the value of the sum

$$\sum_{i=1}^{10} (6i + 2i^2).$$

- [ ] 1230
- [ ] 940
- [ ] 1310
- [ ] 1120
- [x] 1100
- [ ] 1910

Problem 9 (18 points)

Three points in 3D-space are given by

$$P = (2, 3, 2), \quad Q = (3, 2, 4), \quad R = (3, 4, 1).$$

In consequence, we have the following two vectors

$$\overrightarrow{PQ} = (1, -1, 2), \quad \overrightarrow{PR} = (1, 1, -1).$$

Mark the correct answers below.

(a) (2 points). The coordinates of the vector $\overrightarrow{QR}$ are

- [x] $(0, 2, -3)$
- [ ] $(0, -2, -3)$
- [ ] $(1, 2, -1)$
- [ ] $(-1, 2, 1)$

(b) (3 points). The cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ equals

- [ ] $(-2, 3, -2)$
- [ ] $(2, 0, -1)$
- [ ] $(1, 3, 1)$
- [x] $(-1, 3, 2)$

(c) (3 points). The area of the triangle with verticies P, Q and R is

- [ ] $\frac{3}{2}$
- [ ] $\frac{\sqrt{10}}{2}$
- [x] $\frac{\sqrt{14}}{2}$
- [ ] $\sqrt{5}$

(d) (3 points). The line through P and Q has parametric equation

- [ ] $(x, y, z) = (3, 2, 4) + t(1, 1, -1)$
- [ ] $(x, y, z) = (3, 2, 4) + t(1, 0, 1)$
- [ ] $(x, y, z) = (2, 3, 2) + t(3, 2, 4)$
- [x] $(x, y, z) = (2, 3, 2) + t(1, -1, 2)$

(e) (3 points). The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

- [x] $\cos^{-1}(-\frac{\sqrt{2}}{3})$
- [ ] $\cos^{-1}(-\frac{2}{5})$
- [ ] $\frac{\pi}{3}$
- [ ] $\cos^{-1}(-\frac{2}{3})$

(f) (4 points). What is the angle between the plane through P, Q and R and the plane with equation $5x + y + z = 3$?

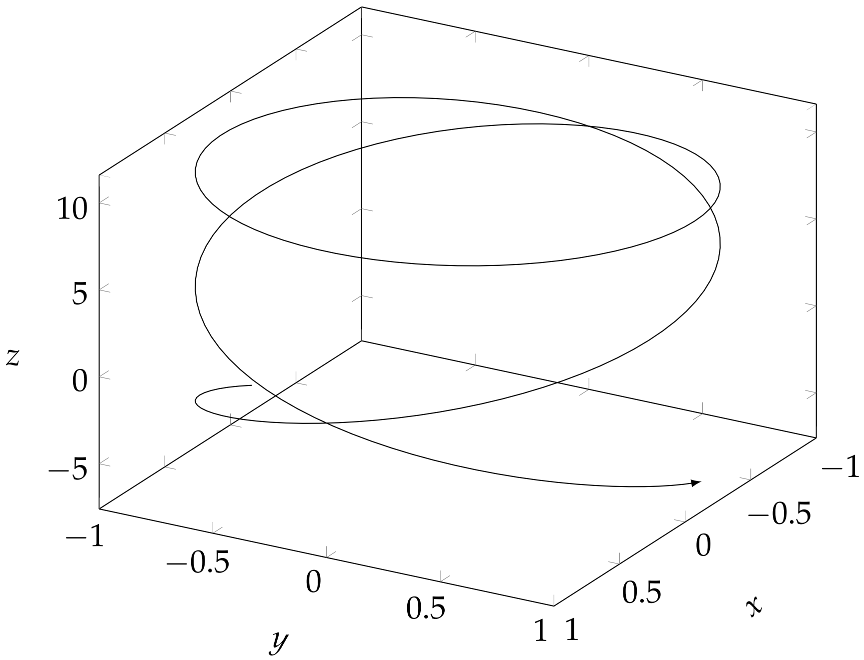
- [ ] $\cos^{-1}(\frac{1}{3})$
- [x] $\frac{\pi}{2}$
- [ ] $\cos^{-1}(\frac{\sqrt{5}}{8})$
- [ ] $\frac{\pi}{3}$

Problem 10 (10 points)

The position vector of a moving particle in 3D-space is given by

$$\vec{r}(t) = \big(\cos(2t), \sin(2t), 10 - t^2\big).$$

Here is a plot of the motion curve when the time t runs from -4 to 4:



(a) (2 points). At time $t = 0$ the particle is located at the point

- ☐ $(1, 1, 10)$
- ☑ $(1, 0, 10)$
- ☐ $(0, -2, 10)$
- ☐ $(1, 1, 0)$
- ☐ $(1, 0, 1)$
- ☐ $(0, -1, 10)$

(b) (4 points). The speed of the particle $v(t)$ is

- ☐ $-\sin(2t) + \cos(2t) + 10 - 2t$
- ☐ 4
- ☑ $2\sqrt{1 + t^2}$
- ☐ $9 - 2t$
- ☐ $\sqrt{1 + 2t}$
- ☐ $\sqrt{10 - 2t}$

(c) (4 points). The velocity vector at time $t = 0$ equals

- ☑ $(0, 2, 0)$
- ☐ $(-1, 1, 0)$
- ☐ $(-2, 2, 0)$
- ☐ $(0, 2, 10)$
- ☐ $(0, 1, 0)$
- ☐ $(0, 1, 10)$

Problem 11 (8 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & 2 & -1 & 5 & 1 \\ 7 & -3 & 2 & 1 & 0 \\ 1 & 0 & -2 & 3 & 2 \\ 2 & 1 & 7 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 1 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}.$$

Mark the correct statements below.

(a) (2 points). The matrix product AB has size

- ☐ 4×5
- ☐ 4×4
- ☑ 4×2
- ☐ 5×2
- ☐ 2×4
- ☐ 3×4

(b) (3 points). Entry $(3, 2)$ of the matrix product AB, i.e. $[AB]_{32}$, equals

- ☐ 3
- ☐ -2
- ☐ 10
- ☑ 6
- ☐ 1
- ☐ 7

(c) (3 points). Entry $(2, 2)$ of the matrix product $B^T B$ i.e. $[B^T B]_{22}$, equals

- ☐ 10
- ☐ -1
- ☐ 16
- ☑ 15
- ☐ 3
- ☐ 21

Problem 12 (6 points)

A matrix is given by

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 0 \\ 4 & 2 & -2 \end{bmatrix}$$

Mark the correct statement below.

- ☐ A is invertible and entry $(1, 3)$ of its inverse, i.e. $[A^{-1}]_{13}$, equals 5.
- ☐ A is invertible and entry $(1, 3)$ of its inverse, i.e. $[A^{-1}]_{13}$, equals -1.
- ☐ A is invertible and entry $(1, 3)$ of its inverse, i.e. $[A^{-1}]_{13}$, equals 7.
- ☑ A is not invertible.
- ☐ None of the above statements apply.

Problem 13 (6 points)

A matrix and a vector are defined by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

One can matrix multiply A by itself n times and get a new matrix A^n. What is the matrix-vector product $A^{100}\vec{v}$?

- ☑ $\begin{bmatrix} 101 \\ 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 100 \\ 100 \end{bmatrix}$
- ☐ $\begin{bmatrix} 201 \\ 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 200 \\ 100 \end{bmatrix}$
- ☐ $\begin{bmatrix} 100 \\ 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 201 \\ 100 \end{bmatrix}$