Answers

Exam in Mathematics for Multimedia Applications

First Year at The Faculty of Engineering and Science

31 May 2016

This exam set consists of 9 pages with 14 problems. For each question a number of points are indicated. The total number of points equals 100.

It is allowed to use books, notes, photocopies etc. It is **not allowed** to use **any electronic devices** such as pocket calculators, mobile phones or computers.

The exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains "multiple choice" problems. **The answers of Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number below. Also write name and student number on each page of your solutions of the essay problems and number these pages. Indicate the total number of extra sheets on the first page.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I (Essay-problems)

Problem 1 (9 points)

(a) (2 points). Prove that the following identity holds:

$$\cos(4x) = \cos^2(2x) - \sin^2(2x).$$

Hint: Use a double angle formula for cosine.

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
Insert $\theta = 2x$.

(b) (4 points). Prove the trigonometric identity

$$\cos(4x) = \cos^4(x) + \sin^4(x) - 6\cos^2(x)\sin^2(x).$$

Hint: Use double angle formulas for sine and cosine.

Insert $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$ above and reduce the expression.

(c) (3 points). Describe all solutions of the equation

$$\cos^4(x) + \sin^4(x) = 6\cos^2(x)\sin^2(x).$$

$x = \frac{\pi}{8} + \frac{\pi}{4}p, \; p \in \mathbb{Z}$

Problem 2 (11 points)

A matrix A and a vector $\mathbf{b}$ are given by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 3 & 8 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) (8 points). Determine whether A is invertible. If so, find its inverse A^{-1}.

(b) (3 points). Solve the equation $A\mathbf{x} = \mathbf{b}$.

(a) A is invertible and $A^{-1} = \begin{bmatrix} 5 & 2 & -2 \\ -2 & -1 & 1 \\ -1 & -2 & 1 \end{bmatrix}$.

(b) $\left\{ \begin{bmatrix} 7 \\ -3 \\ -4 \end{bmatrix} \right\}$

Part II (Multiple-choice problems)

Problem 3 (4 points)

A function is given by

$$f(x) = 3x^2 + \cos(x^2 + x - 1).$$

Mark the correct expression for its derivative $f'(x)$.

- ☐ $6x - \sin(x^2 + x - 1)$
- ☐ $-\sin(x^2 + x - 1)$
- ☐ $6x^2 - \sin(x^2 + x - 1)$
- ☐ $6x^2 + \sin(x^2 + x - 1)$
- ☒ $6x - (2x + 1)\sin(x^2 + x - 1)$
- ☐ $6x - 2x\sin(x^2 + x - 1)$

Problem 4 (5 points)

What is the value of the limit

$$\lim_{h \to 0} \frac{(a+h)^2 + 2(a+h) - (a^2 + 2a)}{h}$$

where a is a constant?

- ☐ -1
- ☐ ∞
- ☐ $a^3 + a^2$
- ☐ $2a$
- ☐ $a^2 + a$
- ☒ $2a + 2$

Problem 5 (6 points)

A function is defined by

$$g(x) = e^{3x} - 2x + 1.$$

The graph of the function has a tangent at the point $(0, 2)$. What is the slope of that tangent?

- ☐ 0
- ☐ 3
- ☒ 1
- ☐ $e - 1$
- ☐ $3e^6 - 2$
- ☐ 2

Problem 6 (3 points)

The sum

$$\sum_{i=1}^{4}(i^2 - 2i)$$

is equal to

- [x] 10
- [ ] 11
- [ ] 3
- [ ] 8
- [ ] -5
- [ ] 13

Problem 7 (5 points)

The sum

$$\sum_{i=1}^{10}(8i^3 + 4i)$$

is equal to

- [ ] 30300
- [ ] 22220
- [ ] 23310
- [x] 24420
- [ ] 11010
- [ ] 21000

Problem 8 (5 points)

A particle is moving along a horizontal axis. Its position as a function of time t is denoted $x(t)$. The velocity of the particle is given by

$$v(t) = 4t + 1$$

and its position at time $t = 0$ is given by $x(0) = 2$. What is the position $x(1)$ of the particle at time $t = 1$?

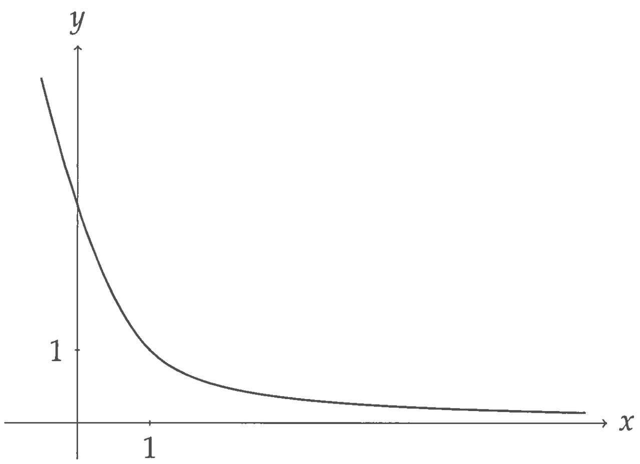
- [ ] 3
- [ ] 4
- [ ] -2
- [ ] 3.5
- [ ] 7
- [x] 5

Problem 9 (8 points)

A function is defined by

$$f(x) = \begin{cases} 3 - 3x + x^2, & x < 1, \\ \frac{1}{x}, & x \geq 1. \end{cases}$$

The graph of the function looks as follows:



(a) (3 points). The integral $\int_1^5 f(x)dx$ equals

- ☐ -3
- ☒ $\ln(5)$
- ☐ 2
- ☐ $2\ln(3)$
- ☐ $\ln(4)$
- ☐ 5

(b) (3 points). The integral $\int_{-1}^1 f(x)dx$ equals

- ☐ $\frac{13}{2}$
- ☒ $\frac{20}{3}$
- ☐ $\frac{11}{2}$
- ☐ $\frac{22}{3}$
- ☐ $\frac{\pi}{4}$
- ☐ 7

(c) (2 point). The integral $\int_0^5 f(x)dx$ equals

- ☒ $\frac{11}{6} + \ln(5)$
- ☐ $3 + 2\ln(3)$
- ☐ $2 + \ln(5)$
- ☐ $\frac{13}{5} + 2\ln(3)$
- ☐ $2 + \ln(4)$
- ☐ 15

Problem 10 (15 points)

Three points in 3D-space are given by

$$P = (2,1,1), \quad Q = (1,1,2), \quad R = (4,3,2).$$

In consequence, we have the following two vectors

$$\overrightarrow{PQ} = (-1,0,1), \quad \overrightarrow{PR} = (2,2,1).$$

Mark the correct answers below.

(a) (2 points). The coordinates of the vector $\overrightarrow{QR}$ are

- ☐ $(1,3,2)$
- ☐ $(2,1,0)$
- ☒ $(3,2,0)$
- ☐ $(-1,1,1)$

(b) (3 points). The angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is

- ☐ $\cos^{-1}(-\frac{1}{10})$
- ☐ $\cos^{-1}(\frac{1}{3})$
- ☒ $\cos^{-1}(-\frac{1}{3\sqrt{2}})$
- ☐ $\frac{\pi}{3}$

(c) (3 points). The cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$ equals

- ☒ $(-2,3,-2)$
- ☐ $(2,0,-1)$
- ☐ $(1,3,1)$
- ☐ $(-2,-3,1)$

(d) (3 points). The line through P and Q has parametric equation

- ☐ $(x,y,z) = (2,1,1) + t(2,0,1)$
- ☒ $(x,y,z) = (2,1,1) + t(-1,0,1)$
- ☐ $(x,y,z) = (1,1,2) + t(1,1,1)$
- ☐ $(x,y,z) = (1,1,2) + t(-1,1,1)$

(e) (4 points). The point S in 3D-space has coordinates $(11,9,4)$. The line through P and R intersect the line through Q and S. What are the coordinates of the point of intersection?

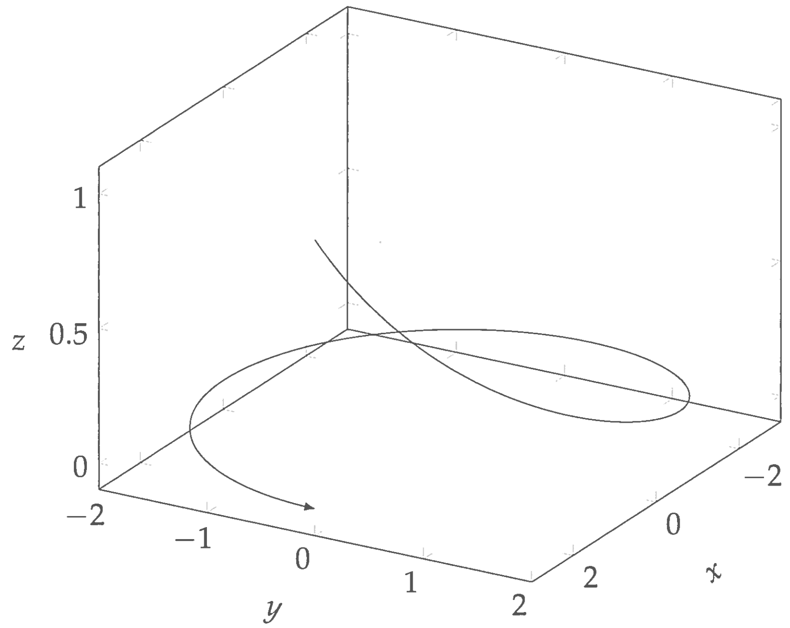
- ☐ $(4,3,2)$
- ☒ $(6,5,3)$
- ☐ $(0,-1,0)$
- ☐ $(1,1,4)$

Problem 11 (10 points)

The position vector of a moving particle in 3D-space is given by

$$\vec{r}(t) = \big(3\cos(t), 2\sin(t), e^{-t}\big).$$

Here is a plot of the motion curve when the time t runs from 0 to 2π:



(a) (2 points). At time $t = 0$ the particle is located at the point

- ☐ $(0, 2, -1)$
- ☒ $(3, 0, 1)$
- ☐ $(0, 3, 1)$
- ☐ $(1, 1, 0)$
- ☐ $(2, 0, 1)$
- ☐ $(3, 2, 0)$

(b) (4 points). The speed of the particle at time $t = 0$ is

- ☒ $\sqrt{5}$
- ☐ 3
- ☐ 2
- ☐ $2 + e$
- ☐ $\sqrt{10}$
- ☐ 2.5

(c) (4 points). The acceleration vector at time $t = 0$ equals

- ☐ $(-3, 0, 0)$
- ☐ $(-1, -1, 0)$
- ☐ $(-3, -2, 1)$
- ☐ $(2, 2, 0)$
- ☐ $(0, -2, 1)$
- ☒ $(-3, 0, 1)$

Problem 12 (7 points)

Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_3 &= 2 \\ x_1 + x_2 + 2x_3 &= 3 \\ 3x_1 + x_2 + 4x_3 &= 7. \end{aligned}$$

Mark the correct statement.

☐ The system is inconsistent.

☐ The system has exactly one solution.

☒ The system has infinitely many solutions.

☐ None of the above statements apply.

Problem 13 (7 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix}.$$

Mark the correct statements below.

(a) (1 point). The matrix product AB has size

☐ 3×4	☐ 2×2	☐ 4×2
☒ 4×4	☐ 2×4	☐ 4×3

(b) (3 points). Entry $(2,1)$ of the matrix product AB, i.e. $[AB]_{21}$, equals

☐ 2	☐ 0	☐ 11
☒ 4	☐ -1	☐ 3

(c) (3 points). Put $C = 2A + B^T$. Entry $(3,1)$ of matrix C, i.e. $[C]_{31}$, equals

☐ 5	☐ -1	☐ 4
☐ -3	☐ 0	☒ 2

Problem 14 (5 points)

Consider the 2×2 rotation matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

One can multiply this matrix by itself and get a new matrix $R_\theta^2 = R_\theta R_\theta$. Which one of the following matrices equals R_θ^2 for any value of the rotation angle θ?

☐ $\begin{bmatrix} \cos(\theta^2) & -\sin(\theta^2) \\ \sin(\theta^2) & \cos(\theta^2) \end{bmatrix}$

☐ $\begin{bmatrix} \cos^2(\theta) & -\sin^2(\theta) \\ \sin^2(\theta) & \cos^2(\theta) \end{bmatrix}$

☒ $\begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$

☐ $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$