Answers

Exam 2015

Mathematics for Multimedia Applications
Medialogy

2 June 2015

Formalities

This exam set consists of 4 pages, in which there are 8 problems. You are allowed to use books, notes etc. You are *not* allowed to use electronic devices such as calculators, computers or cell phones.

A number of points is indicated for every sub-problem. The sum of these points equals 100.

Date and time for the exam: 2 June, 9:00 - 13:00.

You must indicate the following on each page:

- Full name
- Study number
- Page number

On the first page, you must indicate

- The total number of pages.

Good luck!

Problems

Problem 1.

1.a. (4 points) Differentiate the function $f(x) = \cos(5x) + e^x$. $f'(x) = -5\sin(5x) + e^x$

1.b. (4 points) Differentiate the function $g(x) = \ln(3x^2 + 5)$. $g'(x) = \frac{6x}{3x^2+5}$

1.c. (3 points) The graph of the function $g(x)$ has a horizontal tangent at a point. Find the x-coordinate of that point.

$X = 0$

Problem 2.

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

2.a. (3 points) Prove that the following identity holds: Insert $\alpha = 2x$ and $\beta = x$.

$$\sin(3x) = \sin(2x)\cos(x) + \cos(2x)\sin(x)$$

Hint: Write $3x$ as $2x + x$ and use a trigonometric addition formula.

2.b. (4 points) Prove the following trigonometric identity:

$$\sin(3x) = 3\cos^2(x)\sin(x) - \sin^3(x)$$

Hint: Use the double angle formulas.

Insert $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$ above and reduce the expression.

2.c. (3 points) Describe all solutions of the equation

$$\sin^3(x) = 3\cos^2(x)\sin(x)$$

$x = \frac{\pi}{3}p, \quad p \in \mathbb{Z}$

Problem 3.

3.a. (3 points) Calculate the sum

$$\sum_{i=1}^{5}(i^2 - 3i)$$

10

3.b. (4 points) Calculate the sum

$$\sum_{i=1}^{100}(4i - 2)$$

20 000

Problem 4. Evaluate the following integrals:

4.a. (5 points) $\int_1^2 (\frac{1}{x} + 2x)\mathrm{d}x$ $\ln(2) + 3$

4.b. (5 points) $\int_0^{\pi/6} \cos(3x)\mathrm{d}x$ $\frac{1}{3}$

Problem 5. Let P, Q and R be points in 3D-space with coordinates $(1, 1, 3)$, $(3, 1, 1)$ and $(1, 2, 2)$ respectively.

5.a. (4 points) Find the coordinates of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Show that the dot product $\overrightarrow{PQ} \bullet \overrightarrow{PR}$ is equal to 2.

$\overrightarrow{PQ} = (2, 0, -2)$, $\overrightarrow{PR} = (0, 1, -1)$,
$\overrightarrow{PQ} \bullet \overrightarrow{PR} = 2 \cdot 0 + 0 \cdot 1 + (-2) \cdot (-1) = 2$.

5.b. (2 points) Find parametric equations of the line $\mathcal{L}$ through P and Q.

$(x, y, z) = (1, 1, 3) + t(2, 0, -2)$

5.c. (3 points) Find the angle between the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. $60°$

5.d. (3 points) Compute the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$. $(2, 2, 2)$

5.e. (3 points) Find the area of the triangle with vertices P, Q and R. $\sqrt{3}$

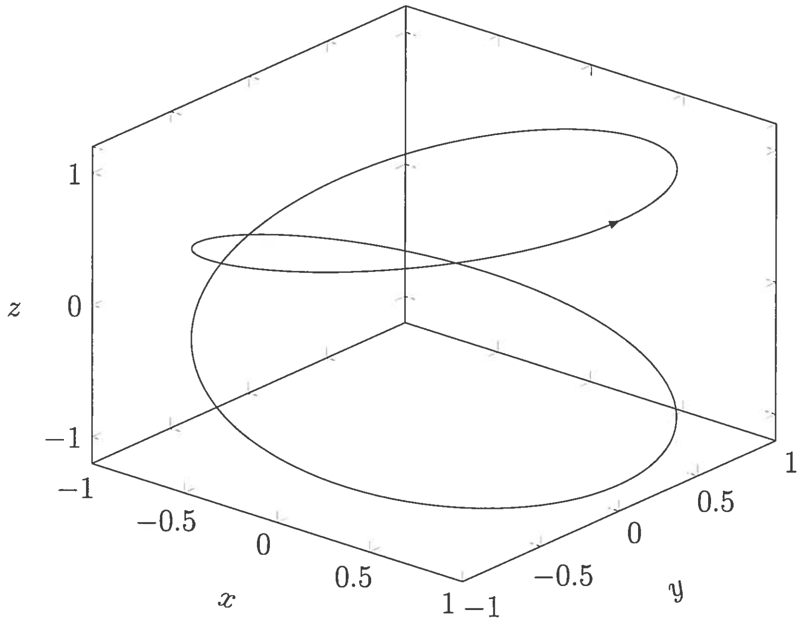
5.f. (3 points) Find an equation of the plane through P, Q and R. $x + y + z = 5$

5.g. (3 points) Find the shortest distance from the point R to the line $\mathcal{L}$. $\frac{\sqrt{6}}{2}$

Problem 6. The position vector of a moving particle in 3D-space is given by

$$\vec{r}(t) = \big(\cos(2t), \sin(2t), \cos(t)\big)$$

Here is a plot of the motion curve when the time t runs from 0 to 2π:



6.a. (3 points) Compute the velocity vector $\vec{v}(t)$. $\vec{v}(t) = (-2\sin(2t), 2\cos(2t), -\sin(t))$

6.b. (2 points) Compute the speed $\nu(t)$. $\nu(t) = \sqrt{4 + \sin^2(t)}$

6.c. (3 points) Compute the acceleration vector $\vec{a}(t)$. $\vec{a}(t) = (-4\cos(2t), -4\sin(2t), -\cos(t))$

6.d. (3 points) Find a unit vector which points in the direction that the particle is moving at time $t = 0$.

$(0, 1, 0)$

7.a. $\left[\begin{array}{cccc|c} 1 & 1 & 5 & 1 & 2 \\ 1 & 2 & 8 & 1 & 4 \\ -1 & 1 & 1 & 0 & 1 \end{array}\right]$

7.b. $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$

Problem 7. Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + 5x_3 + x_4 &= 2 \\ x_1 + 2x_2 + 8x_3 + x_4 &= 4 \\ -x_1 + x_2 + x_3 &= 1 \end{aligned}$$

7.a. (3 points) Find the augmented matrix of the system.

7.b. (6 points) Find the reduced row echelon form of the augmented matrix.

7.c. (4 points) Write down the general solution of the system.

7.d. (3 points) Find a solution of the system which has $x_3 = 1$.

7.c. $\begin{cases} x_1 = 1 - 2x_3 \\ x_2 = 2 - 3x_3 \\ x_3 \text{ free} \\ x_4 = -1 \end{cases}$

7.d. $x_1 = -1$, $x_2 = -1$, $x_3 = 1$, $x_4 = -1$.

Problem 8. Define four matrices as follows:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

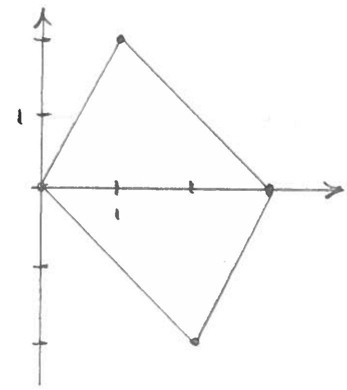
8.a. (4 points) Compute the matrix product AB. $\begin{bmatrix} 1 & 1 \\ 7 & 5 \end{bmatrix}$

8.b. (3 points) Compute $(C + D)^T$. $\begin{bmatrix} 0 & 5 \\ 3 & -4 \end{bmatrix}$

8.c. (4 points) Determine whether C is invertible. If so, find its inverse. C is invertible $C^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

8.d. (3 points) The unit cube in $\mathcal{R}^2$ has vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Sketch the image of this unit cube under the linear transformation

$$T : \mathcal{R}^2 \to \mathcal{R}^2; \quad T(\vec{x}) = D\vec{x}.$$



Appendix

Exact values for trigonometric functions of various angles.

	$0°$	$30°$	$45°$	$60°$	$90°$
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0