Exam 2012

Answers

Mathematics for Multimedia Applications
AAU-Cph, Medialogy

4. June 2012

Formalities

This exam set consists of 5 pages, in which there are 9 problems in total. You are allowed to use books, notes etc. You are *not* allowed to use electronic devices such as calculators, computers or cell phones.

A number of points is indicated for every sub-problem. The sum of these points equals 100.

Date and time for the exam: 4. June, 9:00 - 13:00

You must indicate the following on each page:

- Full name
- Study number
- Page number

On the first page, you must indicate

- The total number of pages.

Good luck!

Problems

Problem 1.

1.a. (3 points) Differentiate the function $f(x) = e^{-2x} + 5e^{3x}$.

$f'(x) = -2e^{-2x} + 15e^{3x}$

1.b. (3 points) Differentiate the function $g(x) = \ln(x^2 + 1)$.

$g'(x) = \frac{2x}{x^2+1}$

1.c. (3 points) Differentiate the function

$$h(x) = \frac{\sin(x)}{x^2}, \quad x \neq 0.$$

$$h'(x) = \frac{\cos(x) \cdot x^2 - \sin(x) \cdot 2x}{(x^2)^2} = \frac{x\cos x - 2\sin x}{x^3}$$

Problem 2. (5 points)
Prove that the following trigonometric identity holds:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta)).$$

Hint: Start with the right hand side of the equation. Use the trigonometric addition formulas.

$\frac{1}{2}(\cos(\alpha - \beta) + \cos(\alpha + \beta)) =$

$\frac{1}{2}(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) = \frac{1}{2} \cdot 2\cos(\alpha)\cos(\beta) =$

$\cos(\alpha)\cos(\beta)$

Problem 3.

3.a. (3 points) Calculate the sum

$$\sum_{i=1}^{4}(2i-5)^2. = (2\cdot1-5)^2 + (2\cdot2-5)^2 + (2\cdot3-5)^2 + (2\cdot4-5)^2$$
$$= (-3)^2 + (-1)^2 + 1^2 + 3^2 = 20$$

3.b. (4 points) Calculate the sum

$$\sum_{i=1}^{100}(2i+1).$$

$$\sum_{i=1}^{100}(2i+1) = 2\sum_{i=1}^{100} i + \sum_{i=1}^{100} 1 = 2 \cdot \frac{100 \cdot 101}{2} + 100 = 10100 + 100 = 10200$$

Problem 4. In this problem, the following values of the sine and cosine functions may be useful:

z	$-\pi/2$	$-\pi/4$	0
$\sin(z)$	-1	-0.71	0
$\cos(z)$	0	0.71	1

Let $f(x) = \cos(2x)$ and $g(x) = x^3 + 1$.

$F(x) = \frac{1}{2}\sin(2x)$

$G(x) = \frac{1}{4}x^4 + x$

4.a. (4 points) Find antiderivatives F and G for f and g.

4.b. (3 points) Evaluate the integrals $\int_{-\pi/4}^{0} f(x)\mathrm{d}x$ and $\int_0^2 g(x)\mathrm{d}x$.

$\frac{1}{2}$, 6

Let

$$h(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0 \end{cases}$$

where f and g are given as above.

4.c. (3 points) Sketch the graph of $h(x)$ on the interval from $-\pi/4$ to 2.

4.d. (3 points) What is $\int_{-\pi/4}^{2} h(x)\mathrm{d}x$? What is $\int_{-\pi/4}^{2} 4h(x)\mathrm{d}x$?

$\frac{13}{2}$, 26

The following values of a function $v(t)$ are observed:

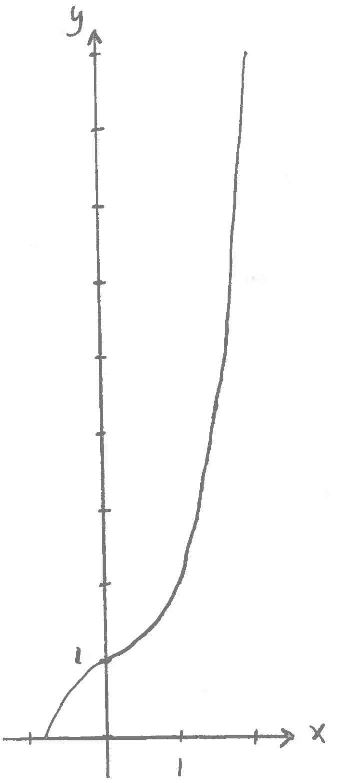
t	-2	0	2	4	6
$v(t)$	3	1	3	9	19

4.e. (3 points) Sketch the graph of $v(t)$.

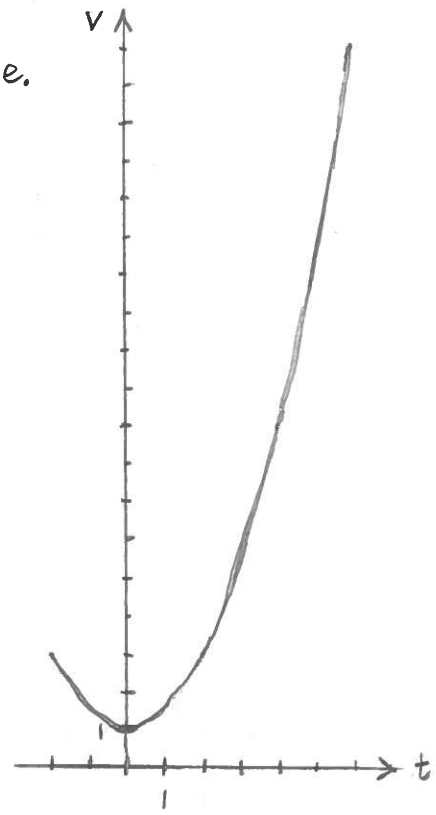
4.f. (5 points) Compute an approximate value of $\int_{-2}^{6} v(t)\mathrm{d}t$ using the trapezoidal approximation.

48

4c.



4.e.



Problem 5. Let P, Q and R be three points in 3D-space. P has coordinates $(-1, 2, 5)$, Q has coordinates $(0, 2, 7)$ and R has coordinates $(2, 1, 10)$.

5.a. (2 points) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. $\quad \overrightarrow{PQ} = (1,0,2)$, $\overrightarrow{PR} = (3,-1,5)$

5.b. (3 points) Compute the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$. $\quad \overrightarrow{PQ} \times \overrightarrow{PR} = (2,1,-1)$

5.c. (3 points) Find the area of the triangle with vertices P, Q and R. $\quad \frac{\sqrt{6}}{2}$

5.d. (2 points) Find a unit vector which is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. $\quad \pm\frac{1}{\sqrt{6}}(2,1,-1)$

5.e. (3 points) Find an equation for the plane through P, Q and R.

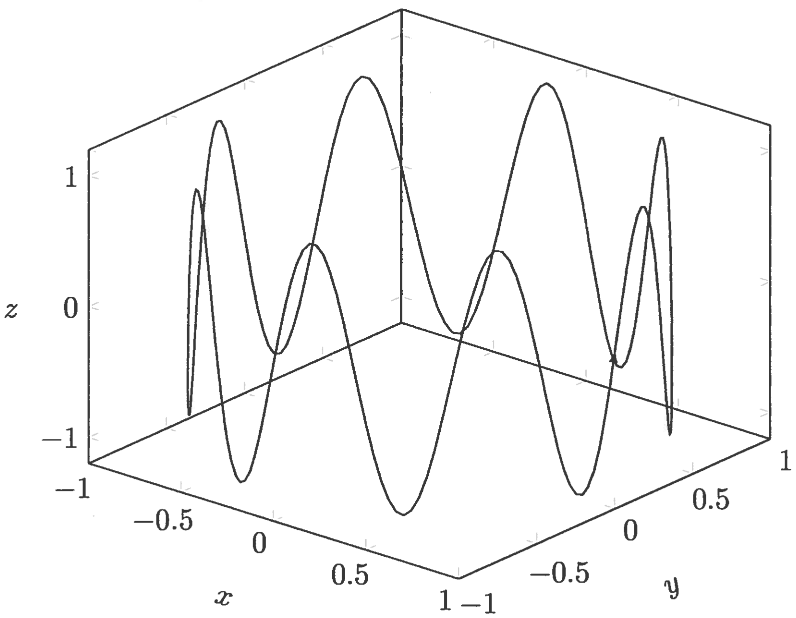
$2\cdot(x-(-1)) + 1\cdot(y-2) + (-1)\cdot(z-5) = 0 \Leftrightarrow$

$2x + y - z = -5$

Problem 6. A parametric curve is given by the following vector function:

$$\vec{r}(t) = (\cos(t), \sin(t), \sin(8t)).$$

Here is a plot of the curve when t runs from 0 to 2π:



6.a. (3 points) Compute the velocity vector $\vec{v}(t)$. $\quad = (-\sin(t), \cos(t), 8\cos(8t))$

6.b. (3 points) Compute the acceleration vector $\vec{a}(t)$. $\quad = (-\cos(t), -\sin(t), -64\sin(8t))$

6.c. (2 points) Find $\vec{r}(0)$ and $\vec{v}(0)$. $\quad (1,0,0)$ and $(0,1,8)$

6.d. (3 points) Find parametric equations for the tangent line to the curve at the point with position vector $\vec{r}(0)$.

$(x,y,z) = (1,0,0) + s(0,1,8)$, $s \in \mathbb{R}$

Problem 7. Consider the following system of linear equations:

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$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ 3x_2 - x_3 &= 0 \\ 2x_1 + x_2 + 7x_3 &= 4 \end{aligned}$$

7.a. (2 points) Is $x_1 = -9$, $x_2 = 1$, $x_3 = 3$ a solution to the system? Why/why not?

7.b. (2 points) Find the augmented matrix of the system.

7.c. (3 points) Find a row echelon form of the augmented matrix.

7.d. (3 points) Find the reduced row echelon form of the augmented matrix.

7.e. (4 points) Write down the general solution to the system.

7.f. (3 points) Find a solution to the system which has $x_3 = 6$.

Problem 8. Define three matrices as follows:

$$A = \begin{bmatrix} 2 & 7 & -1 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

8.a. (3 points) Compute the matrix product AB.

$AB = \begin{bmatrix} 8 & 5 \\ 6 & 5 \end{bmatrix}$

8.b. (2 points) Compute $A^T + B$.

$A^T + B = \begin{bmatrix} 3 & 5 \\ 8 & 2 \\ 0 & 0 \end{bmatrix}$

8.c. (3 points) Find $B^T A^T$.

$B^T A^T = \begin{bmatrix} 8 & 6 \\ 5 & 5 \end{bmatrix}$

8.d. (3 points) Compute C^{-1}.

$C^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

Problem 9. Consider the unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ in an (x, y)-coordinate system. Sketch the image of this square under the following matrix transformations:

9.a. (2 points) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

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9.b. (2 points) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

9.c. (2 points) $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

7a. Yes , insert

7.b. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 3 & -1 & 0 \\ 2 & 1 & 7 & 4 \end{array}\right]$

7.c. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

7.d. $\left[\begin{array}{ccc|c} 1 & 0 & \frac{11}{3} & 2 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

7.e. $\begin{cases} x_1 = 2 - \frac{11}{3} x_3 \\ x_2 = \frac{1}{3} x_3 \\ x_3 \text{ free} \end{cases}$

7.f. $x_1 = -20$, $x_2 = 2$, $x_3 = 6$