20. Session: The Inverse of a Matrix

Recall: Matrix multiplication

A $m\times n$-matrix, B $n\times p$-matrix

$A\cdot B$ is the $m\times p$-matrix with entries

$$[A\cdot B]_{ij} = \sum_{k=1}^{n} [A]_{ik}\cdot[B]_{kj}.$$

Ex.: $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\cdot\begin{bmatrix} 5 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -1 & -3 \end{bmatrix}.$

Recall: Identity matrix I_n

$$I_1 = [1],\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ldots$$

Recall:
- $A\cdot(B\vec{v}) = (A\cdot B)\vec{v}$, $\quad\vec{v}$ $p\times 1$-vector.
- $I_m\cdot A = A\cdot I_n = A.$

! Def. An $n\times n$-matrix A is called invertible if there exists an $n\times n$-matrix B such that

$$A\cdot B = I_n \quad\text{and}\quad B\cdot A = I_n.$$

In this case, B is called the inverse of A and denoted $B = A^{-1}$.

Note: The inverse of an invertible matrix is unique: If both B and C are inverses of A, then

$$B = BI_n = B\cdot(AC) = (BA)C = I_nC = C.$$

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\cdot\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\cdot\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Thus, $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is invertible and $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$

Note: If A is invertible, then

$$A^{-1} \cdot A \cdot \vec{v} = \vec{v} \quad \text{and} \quad A \cdot A^{-1} \cdot \vec{v} = \vec{v}.$$

This follows from $A^{-1}A = AA^{-1} = I_n$ and $I_n \vec{v} = \vec{v}$. Thus,

$$\vec{v} \xmapsto{A\cdot} A\vec{v} \xmapsto{A^{-1}\cdot} \vec{v} \quad \text{and} \quad \vec{v} \xmapsto{A^{-1}\cdot} A^{-1}\vec{v} \xmapsto{A\cdot} \vec{v}.$$

Note: If A is invertible, then

$$A\vec{x} = \vec{b} \Leftrightarrow \vec{x} = A^{-1}\vec{b}$$

Proof: $\Rightarrow$) $A^{-1}\cdot$, $\Leftarrow$) $A\cdot$ q.e.d.

Ex.: $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 5x_2 = -2 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \overset{\text{Ex. above}}{\Leftrightarrow}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix}.$$

Theorem: Let A and B be invertible matrices. Then,

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(2) $A \cdot B$ is invertible and $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

(3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof of (2):

$$AB\,B^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

$$B^{-1}A^{-1}AB = B^{-1}I_nB = B^{-1}B = I_n. \qquad \text{q.e.d.}$$

An Algorithm for Matrix Inversion

! **Theorem:** Let A and B be $n \times n$-matrices. If $A \cdot B = I_n$, then A and B are both invertible and $A^{-1} = B$, $B^{-1} = A$.

(no proof)

Problem: Let A be an $n \times n$-matrix. Determine whether A is invertible. If so, find its inverse A^{-1}.

Idea: Solve the equation $A \cdot X = I_n$.

$X = [\vec{x}_1 \; \vec{x}_2 \; \dots \; \vec{x}_n]$ unknown $n\times n$-matrix and

$I_n = [\vec{e}_1 \; \vec{e}_2 \; \dots \; \vec{e}_n]$. We have

$$AX = I_n \Leftrightarrow [A\vec{x}_1 \; A\vec{x}_2 \; \dots \; A\vec{x}_n] = [\vec{e}_1 \; \vec{e}_2 \dots \vec{e}_n] \Leftrightarrow$$

$$A\vec{x}_1 = \vec{e}_1, \; A\vec{x}_2 = \vec{e}_2, \dots, A\vec{x}_n = \vec{e}_n.$$

We must solve n systems of linear equations:

$[A \mid \vec{e}_1]$, $[A \mid \vec{e}_2]$, ..., $[A \mid \vec{e}_n]$.

We use Gaussian elimination on the $n \times 2n$-matrix

$[A \mid I_n]$.

! Algorithm for Matrix Inversion

Let A be an $n\times n$-matrix. Form the $n\times 2n$-matrix $[A \mid I_n]$. If $[A \mid I_n]$ can be row reduced into a matrix of the form

$$[I_n \mid C],$$

for some $n\times n$-matrix C, then A is invertible and $A^{-1} = C$. Otherwise, A is not invertible.

Ex. $A = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -6 & 0 & 1 \end{array}\right] \xrightarrow{2r_1 + r_2 \to r_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Thus, A is not invertible.

Ex. $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{-2r_1 + r_2 \to r_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{-3r_2 + r_1 \to r_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Thus, A is invertible, and $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

Ex.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{2r_2+r_3 \to r_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array}\right] \xrightarrow{-r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \xrightarrow{-3r_3+r_1 \to r_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \xrightarrow{-2r_2+r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right].$$

Thus, A is invertible, and

$$A^{-1} = \begin{bmatrix} -16 & 4 & 3 \\ -2 & 1 & 0 \\ 7 & -2 & -1 \end{bmatrix}.$$