19. Session : Matrix Multiplication

Recall : (Matrix - vector product)

$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$ $\quad$ $m \times n$ - matrix

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\quad$ $n \times 1$ - vector

$A \cdot \vec{v} = v_1 \vec{a}_1 + v_2 \vec{a}_2 + \dots + v_n \vec{a}_n$ $\quad$ $m \times 1$ - vector.

Ex. $\begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + (-2) \cdot 7 + 4 \cdot (-1) \\ 1 \cdot 0 + (-2) \cdot 2 + 4 \cdot 0 \\ 1 \cdot 3 + (-2) \cdot 0 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} -17 \\ -4 \\ 7 \end{bmatrix}$

! Def.: (Matrix multiplication)

A $\quad$ $m \times n$ - matrix

$B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p]$ $\quad$ $n \times p$ - matrix

The product matrix $A \cdot B$ is defined by

$A \cdot B = [A \cdot \vec{b}_1 \ \ A \cdot \vec{b}_2 \ \dots \ A \cdot \vec{b}_p]$ $\quad$ $m \times p$ - matrix.

Ex. $\begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & 1 \end{bmatrix} = \left[\begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \ \ \begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right] =$

$\begin{bmatrix} -17 & 2 \\ -4 & 0 \\ 7 & 10 \end{bmatrix}$ $\quad$ "$(3 \times 3) \cdot (3 \times 2) = (3 \times 2)$"

! Note : "$(m \times n) \cdot (q \times p) = (m \times p)$"

$n = q$

The product is not defined if $n \neq q$.

Theorem : (The row-column rule)

Let A be an $m \times n$ - matrix and B an $n \times p$ - matrix.

Then the (i,j) - entry in $A \cdot B$ is

$$[A \cdot B]_{ij} = \sum_{k=1}^{n} [A]_{ik} \cdot [B]_{kj}.$$

!Ex.

2. column   (1,2)-entry: $1\cdot 3 + 7\cdot 0 + (-1)\cdot 1 = 2$

1. row

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -17 & 2 \\ -4 & 0 \\ 7 & 10 \end{bmatrix}$$

$$(3\times 3)\cdot(3\times 2) \qquad (3\times 2)$$

Ex.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1)+2\cdot 3 & 1\cdot 1 + 2\cdot 2 \\ 3\cdot(-1)+4\cdot 3 & 3\cdot 1 + 4\cdot 2 \\ 5\cdot(-1)+6\cdot 3 & 5\cdot 1 + 6\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 9 & 11 \\ 13 & 17 \end{bmatrix}$$

$$(3\times 2)\cdot(2\times 2) \qquad (3\times 2)$$

Theorem: For any $m\times n$-matrix A, $n\times p$-matrix B and $p\times 1$-vector $\vec{v}$, one has
$$(A\cdot B)\cdot\vec{v} = A\cdot(B\cdot\vec{v}).$$

Proof: Recall that
$$A\cdot(\vec{u}+\vec{w}) = A\vec{u} + A\vec{w} \quad \text{and} \quad A\cdot(c\vec{u}) = c(A\vec{u}).$$
For $B = [\vec{b}_1\ \vec{b}_2\ \dots\ \vec{b}_p]$ and $\vec{v} = [v_1\ v_2\ \dots\ v_p]^T$, we have $B\vec{v} = v_1\vec{b}_1 + v_2\vec{b}_2 + \dots + v_p\vec{b}_p$. Thus,
$$\begin{aligned} A\cdot(B\vec{v}) &= A\cdot(v_1\vec{b}_1 + v_2\vec{b}_2 + \dots + v_p\vec{b}_p) \\ &= A\cdot(v_1\vec{b}_1) + A\cdot(v_2\vec{b}_2) + \dots + A\cdot(v_p\vec{b}_p) \\ &= v_1(A\vec{b}_1) + v_2(A\vec{b}_2) + \dots + v_p(A\vec{b}_p) \\ &= [A\vec{b}_1\ A\vec{b}_2\ \dots\ A\vec{b}_p]\cdot\vec{v} \\ &= (AB)\cdot\vec{v} \end{aligned}$$
q.e.d.

$$\begin{tikzcd} & B\vec{v} \arrow[dr, "A\cdot"] & \\ \vec{v} \arrow[ur, "B\cdot"] \arrow[rr, "AB\cdot"] & & (AB)\vec{v} = A(B\vec{v}) \end{tikzcd}$$

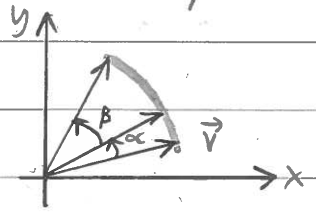
!Remark: In general, $A\cdot B \neq B\cdot A$.
If $AB = BA$, we say that A and B commute.

Ex. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Ex. Recall: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, 2×2-rotation matrix.

$$(R_\alpha R_\beta)\vec{v} = R_{\alpha+\beta} \cdot \vec{v} = R_{\beta+\alpha} \cdot \vec{v} = (R_\beta R_\alpha)\vec{v}.$$



The identity holds for $\vec{v} = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so we have $R_\alpha \cdot R_\beta = R_\beta \cdot R_\alpha$.

Thus R_α and R_β commute.

Remark: $A \cdot B = 0$ does not imply that $A = 0$ or $B = 0$.

Ex. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

Recall: The $n \times n$-identity matrix I_n:

$I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

Theorem: For matrices A, B, C, P, Q of appropriate sizes, one has:

(1) $s(AB) = (sA)B = A(sB)$ for any scalar s

(2) $A(CP) = (AC)P$

(3) $(A+B)C = AC + BC$

(4) $C(P+Q) = CP + CQ$

(5) $I_k A = A = A I_m$

(6) $(AC)^T = C^T A^T$.

Proof of (6): If A is a $k \times m$-matrix and C an $m \times n$-matrix, then $(AC)^T$ and $C^T A^T$ are both $n \times k$-matrices OK. We check that they also have the same entries:

$$[(AC)^T]_{ij} = [AC]_{ji} = \sum_{l=1}^{m} [A]_{jl} \cdot [C]_{li}$$

$$= \sum_{l=1}^{m} [C]_{li} \cdot [A]_{jl} = \sum_{l=1}^{m} [C^T]_{il} \cdot [A^T]_{lj}$$

$$= [C^T A^T]_{ij}$$ OK q.e.d.

Special 2×2-matrices

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ reflection through x-axis

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$ reflection through y-axis

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ reflection through the line $y = x$.

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$ reflection through the line $y = -x$

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$ scaling

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$ horizontal shear

$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$ vertical shear

and the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$