17. Session: Gaussian Elimination

Recall: The elementary row operations

- $r_i \leftrightarrow r_j$ interchange
- $cr_j \to r_j$, $c \neq 0$ scaling
- $kr_i + r_j \to r_j$, $i \neq j$ row addition

Recall: Row echelon form (REF)

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \blacksquare & * & * & \cdots & * & * & * & \cdots & * & * & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \blacksquare & * & * & \cdots & * & * & * & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \blacksquare & * & * & & \\ & & & & & & & & & & & & & \ddots & \end{bmatrix}$$

$*$ any number

$\blacksquare$ nonzero number

Reduced row echelon form (RREF)

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & * & * & \cdots & * & 0 & * & * & \cdots & * & 0 & * & * \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & * & * & \cdots & * & 0 & * & * & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & * & * & & & & \\ & & & & & & & & & & & & & \ddots & & & \end{bmatrix}$$

Gaussian Elimination

Recall: A given matrix can be transformed into one and only one matrix in RREF by means of a sequence of elementary row operations.

Gaussian Elimination is an algorithm which computes the RREF of a matrix via elementary row operations.

Def. Let R be the RREF of a matrix A.

- The position that contains the first nonzero entry in a nonzero row of R (the pivot) is called a pivot position of A
- A column of A that contains some pivot position is called a pivot column.

[Link : The Row Reduction Algorithm]

Ex.

$$\begin{bmatrix} 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 6 & 0 & -6 & 5 & 16 & 7 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 6 & 0 & -6 & 5 & 16 & 7 \end{bmatrix} \xrightarrow{-6r_1 + r_3 \to r_3}$$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ \hline 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 6 & -12 & -13 & 10 & 13 \end{bmatrix} \xrightarrow{-3r_2 + r_3 \to r_3} \underbrace{\begin{bmatrix} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 2 & 4 & -2 \end{bmatrix}}_{\text{in REF}} \xrightarrow{\frac{1}{2}r_3 \to r_3}$$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & -4 & -5 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow[-3r_3 + r_1 \to r_1]{5r_3 + r_2 \to r_2} \begin{bmatrix} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 2 & -4 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2 \to r_2}$$

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & -5 & 2 \\ 0 & 0 & 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} \xrightarrow{r_2 + r_1 \to r_1} \underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}}_{\text{in RREF}}$$

From the RREF to the general solution

Ex.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \end{array} \qquad \begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 3 \end{cases}$$

Ex. (REF)

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 5 & 8 & 7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right] \end{array} \qquad 0x_1 + 0x_2 + 0x_3 = -1$$

no solution

Ex.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 7 & 5 & 8 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{cases} x_1 + 7x_3 + 5x_4 = 8 \\ x_2 + 2x_3 - x_4 = 2 \end{cases}$$

x_1 and x_2 are basic variables (pivot below)

x_3 and x_4 are free variables

Solve the system for the basic variables in terms of the free variables

$$\begin{cases} x_1 = 8 - 7x_3 - 5x_4 \\ x_2 = 2 - 2x_3 + x_4 \\ x_3 \text{ free} \\ x_4 \text{ free} \end{cases}$$

Procedure for solving a system of linear equations

1. Write the augmented matrix $[A \mid \vec{b}]$ of the system.
2. Find the RREF $[R \mid \vec{c}]$ of $[A \mid \vec{b}]$ (use eg. Gaussian elimination).
3. If $[R \mid \vec{c}]$ has a row of the form $[0 \ 0 \ \dots \ 0 \mid 1]$, then the system has no solution. Otherwise, the system has at least one solution. Write the system corresponding to $[R \mid \vec{c}]$, and solve it for the basic variables in terms of the free variables.

Def. A system of linear equations that has one or more solutions is called consistent; otherwise, the system is called inconsistent.

Ex. Solve the system

$$
\begin{aligned}
3x_1 + 5x_2 - 4x_3 &= 7 \\
-3x_1 - 2x_2 + 4x_3 &= -1 \\
6x_1 + x_2 - 8x_3 &= -4 .
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right]
\xrightarrow[-2r_1 + r_3 \to r_3]{r_1 + r_2 \to r_2}
\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right]
\xrightarrow{3r_2 + r_3 \to r_3}
$$

$$
\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]
\xrightarrow{\frac{1}{3} r_2 \to r_2}
\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]
\xrightarrow{-5r_2 + r_1 \to r_1}
$$

$$
\left[\begin{array}{ccc|c} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]
\xrightarrow{\frac{1}{3} r_1 \to r_1}
\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}
$$

$$\begin{cases} x_1 - \frac{4}{3}x_3 = -1 \\ x_2 = 2 \end{cases}$$

$$\begin{cases} x_1 = -1 + \frac{4}{3}x_3 \\ x_2 = 2 \\ x_3 \text{ free} \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}t \\ 2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

The solution set is a line in 3D-space.