16. Session: Systems of Linear Equations

<u>Def.</u> A <u>linear equation</u> in the variables (unknowns) $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b,$$

where $a_1, a_2, \dots, a_n$ and b are real numbers.

<u>Ex.</u> $5x_1 + 7x_2 + (-1)x_3 = 2$

<u>Ex.</u> $3x_1 + x_3 = 5 \Leftrightarrow 3x_1 + 0x_2 + 1x_3 = 5$

<u>Ex.</u> $3x_1^2 + x_2 - x_3 = 7$ not linear

<u>Ex.</u> $5x_1 - 2x_2 + x_3 = 4x_1 + 3 \Leftrightarrow 1x_1 + (-2)x_2 + 1x_3 = 3$

<u>Def.</u> A <u>system of linear equations</u> is a set of m equations in the same n variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

The <u>augmented matrix</u> of the system is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{A} \quad \underbrace{\quad}_{\vec{b}}$$

The submatrix A is called the <u>coefficient matrix</u>.

<u>Note:</u> The system can be written as $A \cdot \vec{x} = \vec{b}$, where $\vec{x} = [x_1 \ x_2 \ \dots \ x_n]^T$.

Ex. $2x_1 + x_2 - x_3 = 3$
$x_1 + 2x_2 + x_3 = 3$
$3x_1 + x_2 = 2$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{array}\right]$$

Ex. $5x_1 + 3x_2 + 4x_3 - x_4 = 0$
$x_1 + 3x_3 + 7x_4 = 1$

$$\left[\begin{array}{cccc|c} 5 & 3 & 4 & -1 & 0 \\ 1 & 0 & 3 & 7 & 1 \end{array}\right]$$

Ex. $3x_1 + 2x_2 = 1$
$-x_1 + 7x_2 = 3$

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ -1 & 7 & 3 \end{array}\right]$$

Def. A solution to a system of linear equations in the variables $x_1, x_2, \dots, x_n$ is a vector $[s_1 \; s_2 \; \dots \; s_n]^T$ such that every equation is satisfied, when we insert

$$x_1 = s_1, \; x_2 = s_2, \dots, x_n = s_n.$$

The solution set of the system is the set of all solutions.

Ex. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a solution to the system $\begin{array}{l} 5x_1 - x_2 = 2 \\ 2x_1 - 3x_2 = -7. \end{array}$

We will now describe a solution procedure via so-called row operations on the augmented matrix.

Notation: r_i denotes row no. i

! **Def.** The elementary row operations are
- $r_i \leftrightarrow r_j$ interchange
- $cr_j \rightarrow r_j$, $c \neq 0$ scaling
- $kr_i + r_j \rightarrow r_j$, $i \neq j$ row addition

Ex.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{array}\right] \xrightarrow{-2r_1 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -3 \\ 3 & 1 & 0 & 2 \end{array}\right] \xrightarrow{\left(-\frac{1}{3}\right) r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{array}\right]$$

Remark: The elementary row operations preserve the solution set of the system of linear equations.

Strategy: Use elementary row operations to simplify the system until it is easy to see what the solutions are.

Def. A matrix is said to be in row echelon form (REF) if

1. Each nonzero row lies above every zero row.
2. The first nonzero entry (from the left) in every row lies to the right of the first nonzero entry in the row above it.

Ex. $\begin{bmatrix} 2 & 5 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 3 & -6 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 5 & 0 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Def. A first nonzero entry in in a row of a matrix in REF is called a pivot.

Def. A matrix is said to be in reduced row echelon form (RREF) if

1. The matrix is in REF.
2. All pivots are equal to 1.
3. All entries above the pivot positions are equal to 0.

Ex. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Theorem: A given matrix can be transformed into one and only one matrix in RREF by means of a sequence of elementary row operations.

Next session:

- How do we find such a sequence?
- How does the RREF give us the solution set?

Ex.

$$\begin{aligned} 5x_1 + 2x_2 + 9x_3 &= 0 \\ 2x_1 + 3x_2 - x_3 &= 9 \\ x_1 - x_2 - 2x_3 &= 1 \end{aligned}$$

Solve this system.

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 5 & 2 & 9 & 0 \\ 2 & 3 & -1 & 9 \\ 1 & -1 & -2 & 1 \end{array}\right] \end{array} \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 2 & 3 & -1 & 9 \\ 5 & 2 & 9 & 0 \end{array}\right] \xrightarrow[-5r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 5 & 3 & 7 \\ 0 & 7 & 19 & -5 \end{array}\right] \xrightarrow{-r_2 + r_3 \to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 5 & 3 & 7 \\ 0 & 2 & 16 & -12 \end{array}\right] \xrightarrow{\frac{1}{2} r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 5 & 3 & 7 \\ 0 & 1 & 8 & -6 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 8 & -6 \\ 0 & 5 & 3 & 7 \end{array}\right] \xrightarrow{-5r_2 + r_3 \to r_3}$$

in REF

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 8 & -6 \\ 0 & 0 & -37 & 37 \end{array}\right] \xrightarrow{-\frac{1}{37} r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 8 & -6 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[2r_3 + r_1 \to r_1]{-8r_3 + r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{r_2 + r_1 \to r_1}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \end{array} \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= -1 \end{aligned}$$

in RREF

The solution set is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$.