15. Session: Linear Combinations, Matrix-Vector Products and Special Matrices

<u>Recall</u>: Matrices, entries

$$A = \begin{bmatrix} 1 & 7 & 0 & 2 \\ 5 & 0 & -1 & 11 \\ -3 & 10 & 7 & 9 \\ -1 & 8 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

5×4 - matrix

5 rows, 4 columns

$(3,2)$-entry : $a_{32} = [A]_{32} = 10$

<u>Recall</u>: $+$, $-$, multiplication by a scalar

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+0 \\ -3+1 & 0+4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-0 \\ -3-1 & 0-4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -4 & -4 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 & 5 \cdot 2 \\ 5 \cdot (-3) & 5 \cdot 0 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -15 & 0 \end{bmatrix}$$

<u>Recall</u>: Transpose

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 5 & 2 \\ 0 & -1 \end{bmatrix}$$

<u>Recall</u>: Vectors, $\mathbb{R}^n$, standard basis vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$$

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Standard basis for $\mathbb{R}^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$$

<u>Recall</u>: Linear combinations of vectors

$$2 \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

weights or coefficients

Matrix-vector products

Def. Let A be an $m\times n$-matrix with column vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$.

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be an $n\times 1$-vector.

The product $A\cdot\vec{v}$ is the $m\times 1$-vector

$$A\cdot\vec{v} = v_1\vec{a}_1 + v_2\vec{a}_2 + \dots + v_n\vec{a}_n .$$

Ex.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 8 \end{bmatrix} = 7\cdot\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8\cdot\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 7\cdot 1 + 8\cdot 2 \\ 7\cdot 3 + 8\cdot 4 \\ 7\cdot 5 + 8\cdot 6 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

Faster:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 7\cdot 1 + 8\cdot 2 \\ 7\cdot 3 + 8\cdot 4 \\ 7\cdot 5 + 8\cdot 6 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

Ex.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-1)\cdot 2 + 1\cdot 3 + 3\cdot 1 \\ (-1)\cdot 1 + 1\cdot(-2) + 3\cdot 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Identity matrices

Def. The $n\times n$-identity matrix I_n is the $n\times n$-matrix with columns $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

Ex.

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \dots$$

Theorem: $I_n\vec{v} = \vec{v}$ for every $n\times 1$-vector $\vec{v}$.

Proof:

$$I_n\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n = \vec{v} \qquad \text{q.e.d.}$$

Ex. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cdot 1 + b\cdot 0 \\ a\cdot 0 + b\cdot 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$

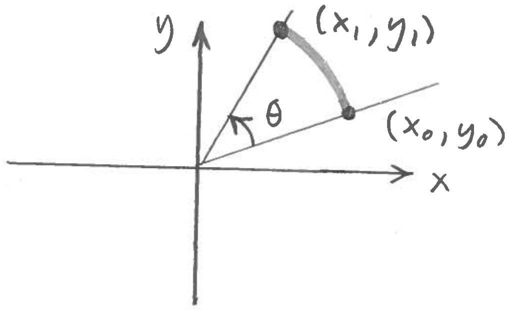
Rotation matrices

! **Def.** For an angle θ, the 2×2-rotation matrix is defined as

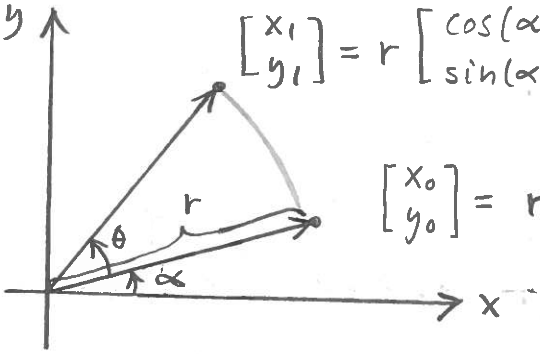
$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Theorem: Let (x_0, y_0) be a point in the plane. The rotation (x_1, y_1) of (x_0, y_0) through the angle θ about the origin, is given by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



Proof.



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = r \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix} = \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = r \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} = \begin{bmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{bmatrix}$$

$$R_\theta \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\alpha)\cos(\theta) - r\sin(\alpha)\sin(\theta) \\ r\cos(\alpha)\sin(\theta) + r\sin(\alpha)\cos(\theta) \end{bmatrix}$$

Addition formulas

$$= \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

q.e.d.

__Theorem__ (Properties of matrix-vector products)
Let A be an $m\times n$-matrix, $\vec{u}$ and $\vec{v}$ vectors in $\mathbb{R}^n$, and c a scalar. Then

(1) $A\cdot(\vec{u}+\vec{v}) = A\cdot\vec{u} + A\cdot\vec{v}$

(2) $A\cdot(c\vec{u}) = c(A\cdot\vec{u})$

(3) $A\cdot\vec{0} = \vec{0}$

__Proof of (1)__

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}.$$

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n].$$

$$\begin{aligned} A\cdot(\vec{u}+\vec{v}) &= (u_1+v_1)\vec{a}_1 + (u_2+v_2)\vec{a}_2 + \dots + (u_n+v_n)\vec{a}_n \\ &= u_1\vec{a}_1 + u_2\vec{a}_2 + \dots + u_n\vec{a}_n + v_1\vec{a}_1 + v_2\vec{a}_2 + \dots + v_n\vec{a}_n \\ &= A\cdot\vec{u} + A\cdot\vec{v} \end{aligned}$$

q.e.d.

__Ex.__

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$