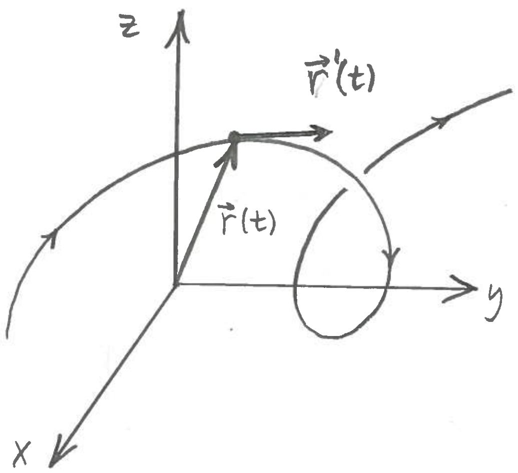
13. Session: Curves and Motion in Space II

Recall: Vector function

$$\vec{r}(t) = (f(t), g(t), h(t)), \quad t \in I$$

Interpretation: Position vector for a moving point, where t is the time.



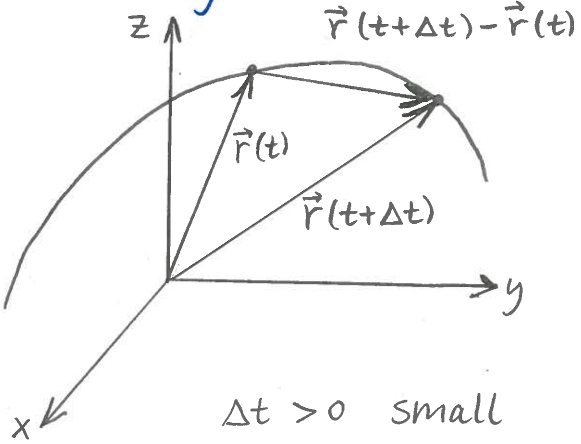
Def.: $\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

Theorem: $\vec{r}'(t) = (f'(t), g'(t), h'(t))$

Note: $\vec{r}'(t)$ tangent vector (when $\vec{r}'(t) \neq \vec{0}$).

Ex. $\vec{r}(t) = (3t^2+1, 2t, 5t^3) \Rightarrow \vec{r}'(t) = (6t, 2, 15t^2)$.

Velocity and Acceleration



Average speed in $[t, t+\Delta t]$:

$$\frac{\|\vec{r}(t+\Delta t) - \vec{r}(t)\|}{\Delta t} \overset{*}{=} \left\| \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \right\|$$

Instantaneous speed at time t:

$$\left\| \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \right\| = \|\vec{r}'(t)\|.$$

! Def. The velocity vector at time t is

$$\vec{v}(t) = \vec{r}'(t).$$

The speed is

$$v(t) = \|\vec{v}(t)\|.$$

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* $k > 0$: $\frac{\|\vec{w}\|}{k} = \frac{1}{k}\|\vec{w}\| = \left|\frac{1}{k}\right| \|\vec{w}\| = \left\|\frac{1}{k}\vec{w}\right\| = \left\|\frac{\vec{w}}{k}\right\|$

Remark:

$\vec{v}(t)$
- direction: tangential to the curve of motion in the direction which the point moves
- length: the speed of the moving point.

!Def. The acceleration vector at time t is

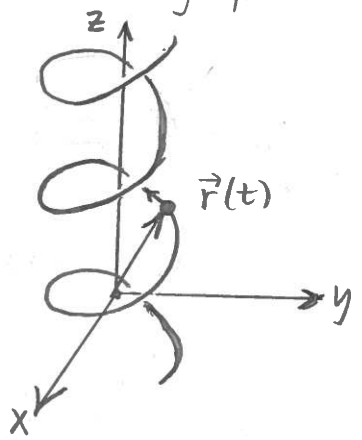
$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t).$$

The scalar acceleration is

$$a(t) = \|\vec{a}(t)\|.$$

Note: In general, $\vec{a}(t) = \vec{v}\,'(t)$ but $a(t) \neq v'(t)$.

Ex.: Moving point on a helix



$$\vec{r}(t) = (\cos(t), \sin(t), t)$$

$$\vec{v}(t) = \vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$$

$$v(t) = \|\vec{v}(t)\| = \sqrt{\underbrace{(-\sin(t))^2 + (\cos(t))^2}_{1} + 1^2}$$

$$= \sqrt{2}$$

$$\vec{a}(t) = \vec{v}\,'(t) = (-\cos(t), -\sin(t), 0)$$

$$a(t) = \|\vec{a}(t)\| = \sqrt{(-\cos(t))^2 + (-\sin(t))^2 + 0^2} = 1$$

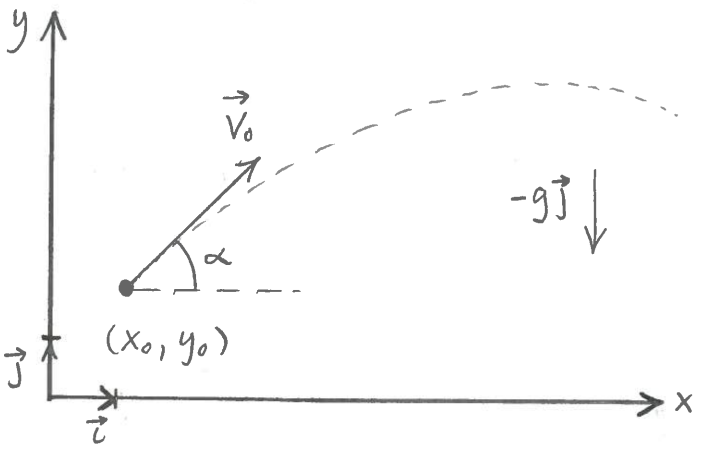
Note that

- The speed and scalar acceleration are constant.
- $a(t) = 1 \neq v'(t) = (\sqrt{2})' = 0$

Note: Since $\vec{a}(t) = \vec{v}\,'(t)$ and $\vec{v}(t) = \vec{r}\,'(t)$, velocity is an antiderivative of acceleration, and position is an antiderivative of velocity

$$\vec{v}(t) = \int \vec{a}(t)\,dt \quad , \quad \vec{r}(t) = \int \vec{v}(t)\,dt.$$

Motion of Projectiles



Launched at time $t=0$.

Initial position $\vec{r}_0 = (x_0, y_0)$

Initial speed $v_0 > 0$.

Angle of inclination: α.

Acceleration due to gravity: $g = 9{,}8 \ \frac{m}{s^2}$.

Initial velocity:

$$\vec{v}_0 = v_0 (\cos\alpha, \sin\alpha) = (v_0 \cos\alpha, v_0 \sin\alpha).$$

We calculate the trajectory of the projectile for $t \geq 0$:

$$\vec{a}(t) = -g\vec{j}$$

$$\vec{v}(t) = \int \vec{a}(t)dt = -gt\vec{j} + \vec{C}_1$$

Since $\vec{v}(0) = \vec{v}_0$, we have $-g \cdot 0 \cdot \vec{j} + \vec{C}_1 = \vec{v}_0 \Rightarrow \vec{C}_1 = \vec{v}_0$, and

$$\vec{v}(t) = -gt\vec{j} + \vec{v}_0.$$

Position vector:

$$\vec{r}(t) = \int \vec{v}(t)dt = -\frac{1}{2}gt^2\vec{j} + \vec{v}_0 t + \vec{C}_2$$

Since $\vec{r}(0) = \vec{r}_0$, we have that $\vec{C}_2 = \vec{r}_0$, and

$$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + \vec{v}_0 t + \vec{r}_0.$$

$$= \left(0, -\frac{1}{2}gt^2\right) + (v_0 \cos(\alpha)t, v_0 \sin(\alpha)t) + (x_0, y_0)$$

$$= \left(v_0 \cos(\alpha)t + x_0, -\frac{1}{2}gt^2 + v_0 \sin(\alpha)t + y_0\right)$$

Parametric equations for the trajectory:

$$x(t) = (v_0 \cos\alpha)\,t + x_0 \,,$$
$$y(t) = -\frac{1}{2} g t^2 + (v_0 \sin\alpha)\,t + y_0 \,, \qquad (t \geqslant 0)$$

Applications

Laws of physics $\rightsquigarrow$ Realistic animations

Ex. [MATLAB program]