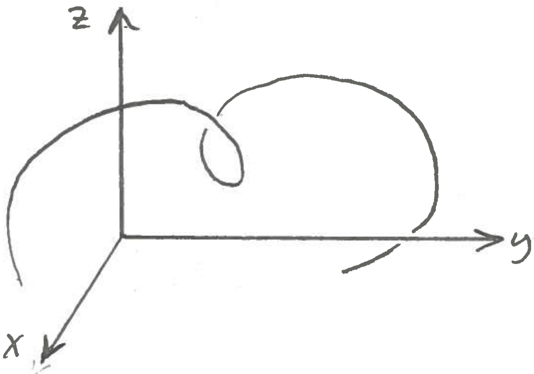
12. Session: Curves and Motion in Space I

Def. A parametric curve in 3D-space consists of three equations

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

where $f(t)$, $g(t)$ and $h(t)$ are continuous functions defined on an interval I. The corresponding curve is the set of points

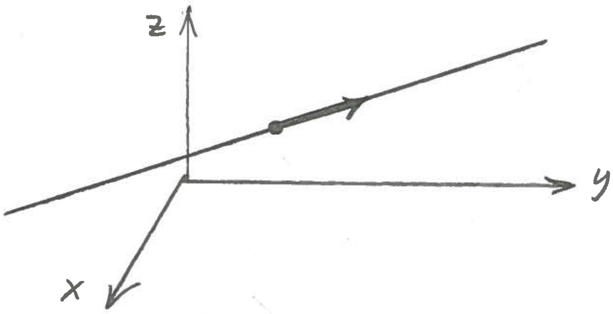
$$\{(x,y,z) \mid x = f(t), y = g(t), z = h(t) \text{ for some } t \in I\}.$$



A curve in 3D-space

Ex. Parametric equations for a line

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

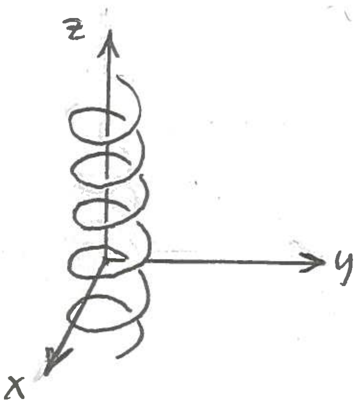


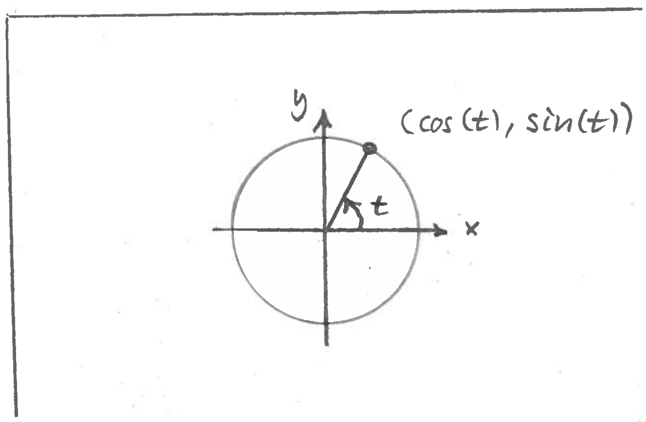
A line in 3D-space

$((a,b,c) \neq \vec{0})$

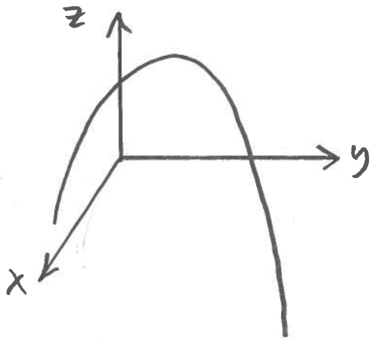
Ex. The helix

$$x = \cos(t), \quad y = \sin(t), \quad z = t$$



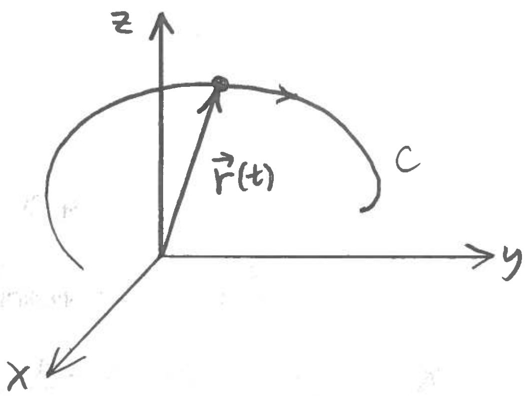


Ex. $x = 2t+1$, $y = 3t$, $z = -t^2 + 5t$



! **Def.** $\vec{r}(t) = (f(t), g(t), h(t))$, $t \in I$ is called a vector function

Geometric interpretation



t is the time.
$\vec{r}(t)$ is the position vector for a point moving along a curve C.

Differentiation of vector functions

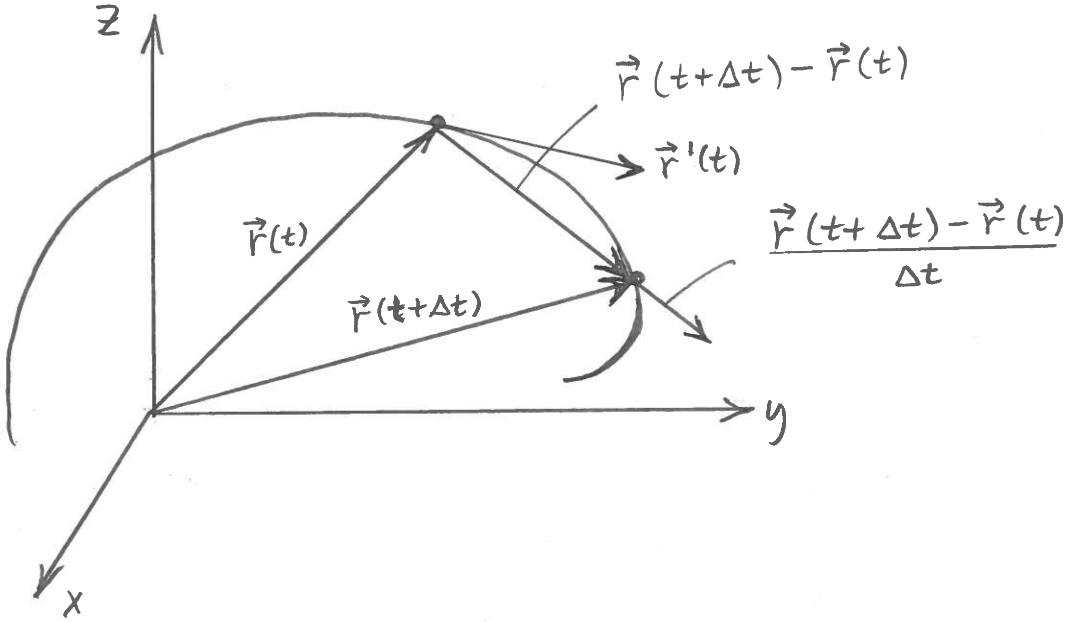
Def. $\lim\limits_{t \to a} \vec{r}(t) = \left(\lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \right)$

Def. The derivative $\vec{r}\,'(t)$ of the vector function $\vec{r}(t)$ is the limit

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

provided that this limit exists.

! **Remark:** If $\vec{r}\,'(t) \neq \vec{0}$, then $\vec{r}\,'(t)$ is a tangent vector to the curve C at the point with position vector $\vec{r}(t)$.



! **Theorem**: If the coordinate functions of $\vec{r}(t) = (f(t), g(t), h(t))$ are differentiable, then

$$\vec{r}\,'(t) = (f'(t), g'(t), h'(t)).$$

Proof:

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)) - (f(t), g(t), h(t))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{f(t+\Delta t)-f(t)}{\Delta t}, \frac{g(t+\Delta t)-g(t)}{\Delta t}, \frac{h(t+\Delta t)-h(t)}{\Delta t} \right)$$

$$= \left(\lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t)-h(t)}{\Delta t} \right)$$

$$= (f'(t), g'(t), h'(t)).$$

q.e.d.

Ex.

$\vec{r}(t) = (2t+1, 3t, -t^2+5t) \Rightarrow$

$\vec{r}\,'(t) = (2, 3, -2t+5)$

Ex. $\vec{r}(t) = (\cos(t), \sin(t), t) \Rightarrow$

$\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$.

For $t=0$ we have

$$\vec{r}(0) = (\cos(0), \sin(0), 0) = (1,0,0)$$

$$\vec{r}\,'(0) = (-\sin(0), \cos(0), 1) = (0,1,1)$$

Thus, $(0,1,1)$ is a tangent vector to the helix at the point $(1,0,0)$.

Theorem: Let $\vec{u}(t)$ and $\vec{v}(t)$ be differentiable vector functions, $f(t)$ a differentiable function and c a scalar. Then,

(1) $(\vec{u}(t) + \vec{v}(t))' = \vec{u}\,'(t) + \vec{v}\,'(t)$

(2) $(c\vec{u}(t))' = c\vec{u}\,'(t)$

(3) $(f(t)\vec{u}(t))' = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

(4) $(\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

(5) $(\vec{u}(t) \times \vec{v}(t))' = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$

Proof of (4):

$\vec{u}(t) = (u_1(t), u_2(t), u_3(t))$, $\vec{v}(t) = (v_1(t), v_2(t), v_3(t))$.

$$\vec{u}(t) \cdot \vec{v}(t) = \sum_{i=1}^{3} u_i(t) v_i(t) \Rightarrow$$

$$(\vec{u}(t) \cdot \vec{v}(t))' = \sum_{i=1}^{3} \left(u_i'(t) v_i(t) + u_i(t) v_i'(t) \right)$$

$$= \sum_{i=1}^{3} u_i'(t) v_i(t) + \sum_{i=1}^{3} u_i(t) v_i'(t)$$

$$= \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$$

q.e.d.