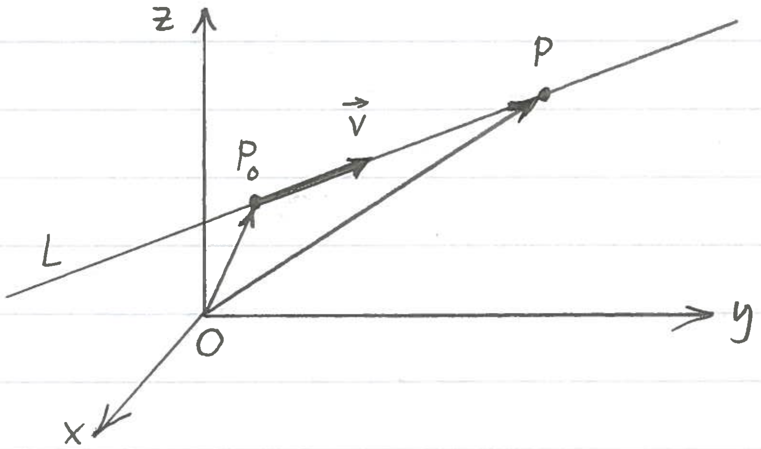
11. Session: Lines and Planes in Space

Notation: $\mathbb{R}$ denotes the set of real numbers.
$\in$ means "belongs to".

Parametric equations for a line

Let $P_0 = (x_0, y_0, z_0)$ be a point and $\vec{v} = (a,b,c) \neq \vec{0}$ a vector in 3D-space. We want to find a description of the line L through P_0 and parallel to $\vec{v}$.



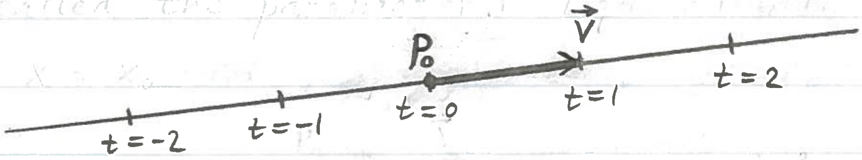
($\vec{v}$ is called a direction vector for the line.)

For any point $P = (x,y,z)$ in 3D-space, we have:

$$P \in L \iff$$
$$\overrightarrow{P_0P} = t\vec{v}, \text{ for some } t \in \mathbb{R} \iff$$
$$\overrightarrow{OP} - \overrightarrow{OP_0} = t\vec{v}, \text{ for some } t \in \mathbb{R} \iff$$
$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{v}, \text{ for some } t \in \mathbb{R} \iff$$
$$(x,y,z) = (x_0, y_0, z_0) + t(a,b,c), \text{ for some } t \in \mathbb{R}.$$

Parametric equation for the line L:

! $$\boxed{(x,y,z) = (x_0, y_0, z_0) + t(a,b,c), \quad t \in \mathbb{R}}$$



(t is called the parameter)

Parametric equations for L:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

Ex. $(x,y,z) = (1,3,-1) + t(1,0,2), \quad t \in \mathbb{R}$

Line through $P_0 = (1,3,-1)$ parallel to $\vec{v} = (1,0,2)$.

Ex. Find a parametric equation for the line through $A = (1,2,2)$ and $B = (3,-1,3)$.

Direction vector:

$\vec{v} = \overrightarrow{AB} = B - A = (3,-1,3) - (1,2,2) = (2,-3,1)$

Point on the line: $P_0 = A = (1,2,2)$.

Parametric equation

$$\underline{\underline{(x,y,z) = (1,2,2) + t(2,-3,1)\ ,\ t \in \mathbb{R}}}$$

Another possibility:

$\vec{v} = 2 \cdot (2,-3,1) = (4,-6,2)$

$P_0 = B = (3,-1,3)$

Parametric equation

$$\underline{\underline{(x,y,z) = (3,-1,3) + t(4,-6,2)\ ,\ t \in \mathbb{R}}}$$

! The parametric equation of a line is not unique.

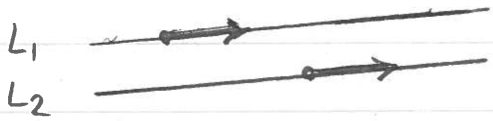
<u>Intersection of lines in 3D-space</u>

$L_1 : \quad (x,y,z) = (x_1,y_1,z_1) + t(a_1,b_1,c_1)\ ,\ t \in \mathbb{R}$

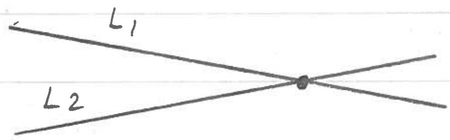
$L_2 : \quad (x,y,z) = (x_2,y_2,z_2) + s(a_2,b_2,c_2)\ ,\ s \in \mathbb{R}$

Note that L_1 and L_2 are parallel if and only if

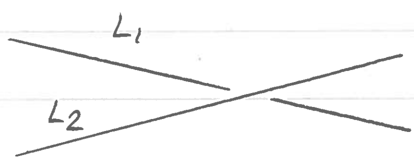
$$(a_1,b_1,c_1) = k(a_2,b_2,c_2) \quad \text{for some } k \in \mathbb{R}.$$



If L_1 and L_2 are not parallel, they are either intersecting or nonintersecting (skew lines):



Intersecting



Skew lines

Procedure:

(1) Solve the system
$$\left.\begin{aligned} x_1 + a_1 t &= x_2 + a_2 s \\ y_1 + b_1 t &= y_2 + b_2 s \end{aligned}\right\}$$

(2) If there is a solution for s and t, which also satisfy $z_1 + c_1 t = z_2 + c_2 s$, then one has a point of intersection.

Symmetric equations for a line

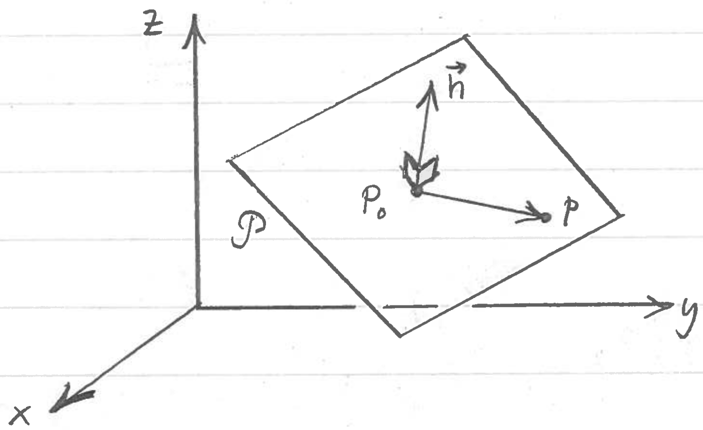
If a, b and c are nonzero, we can eliminate the parameter in the parametric equations:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \iff$$
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

Symmetric equations for the line:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Planes in 3D-space



Let $\mathcal{P}$ be the plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n} = (a, b, c) \neq \vec{0}$.

For any point $P = (x, y, z)$ we have

$$P \in \mathcal{P} \iff$$
$$\vec{n} \cdot \overrightarrow{P_0 P} = 0 \iff$$
$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0 \iff$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Equation for the plane $\mathcal{P}$:

! $\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

Ex. The plane through $P_0 = (-1, 5, 2)$ with normal vector $\vec{n} = (1, -3, 2)$ has equation

$$1 \cdot (x - (-1)) + (-3) \cdot (y - 5) + 2 \cdot (z - 2) = 0 \Leftrightarrow$$

$$\underline{\underline{x - 3y + 2z = -12}}$$

! Theorem: Every equation of the form

$$ax + by + cz = d$$

where $a \neq 0$, $b \neq 0$ or $c \neq 0$, is an equation for a plane with normal vector (a, b, c).

Proof: If for instance $c \neq 0$, we have

$$ax + by + cz = d \Leftrightarrow ax + by + c(z - \tfrac{d}{c}) = 0,$$

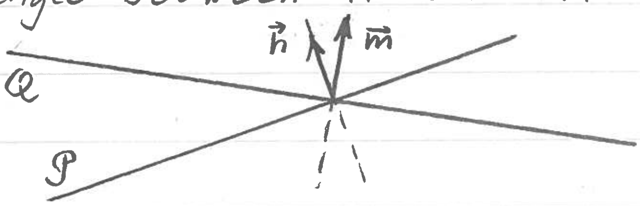
which is the equation for the plane through $(0, 0, \frac{d}{c})$ with normal vector (a, b, c). q.e.d.

Angles between planes

$\mathcal{P}$ plane with normal vector $\vec{n} \neq \vec{0}$,

$\mathcal{Q}$ plane with normal vector $\vec{m} \neq \vec{0}$,

θ angle between $\vec{n}$ and $\vec{m}$.



(seen from the side)

- If $\theta = 0$ or $\theta = \pi$, then $\mathcal{P}$ and $\mathcal{Q}$ are parallel.
- If $\theta \neq 0$ and $\theta \neq \pi$, then the angle between $\mathcal{P}$ and $\mathcal{Q}$ is either θ or $\pi - \theta$, whichever is an acute angle (and the planes intersect in a line).

Ex. $\mathcal{P}: \quad 2x + 3y - z = -3$

$\mathcal{Q}: \quad 4x + 5y + z = 1$

Then we have normal vectors $\vec{n} = (2, 3, -1)$, $\vec{m} = (4, 5, 1)$.

$$\cos(\theta) = \frac{\vec{n} \bullet \vec{m}}{\|\vec{n}\| \|\vec{m}\|} = \frac{2 \cdot 4 + 3 \cdot 5 + (-1) \cdot 1}{\sqrt{2^2 + 3^2 + (-1)^2} \sqrt{4^2 + 5^2 + 1^2}} = \frac{22}{\sqrt{14} \cdot \sqrt{42}} .$$

$\theta \approx 24,87°$ acute OK, so the angle between $\mathcal{P}$ and $\mathcal{Q}$ is $\underline{\underline{24,87°}}$