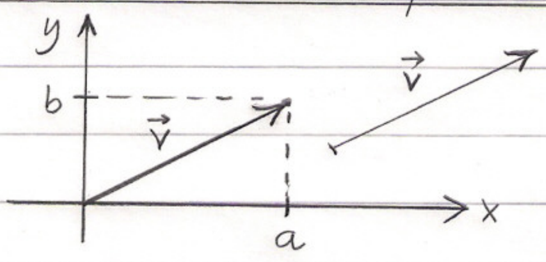
8. Session : Vectors

Def. : A vector in the plane (2D-vector) is an ordered pair of real numbers

$$\vec{v} = (a, b)$$

The numbers a and b are called coordinates or components of $\vec{v}$. A scalar is a real number.

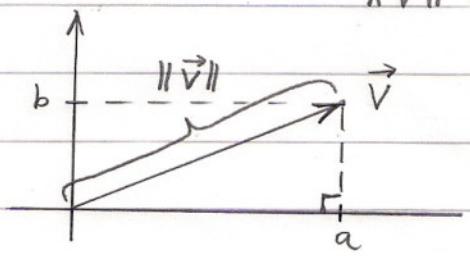
Geometric interpretation



Two representatives for $\vec{v}$.

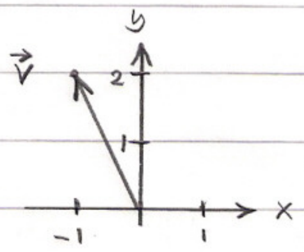
Def. The length of $\vec{v} = (a, b)$ is

$$\|\vec{v}\| = \sqrt{a^2 + b^2}.$$



Ex. $\vec{v} = (-1, 2)$

$\|\vec{v}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

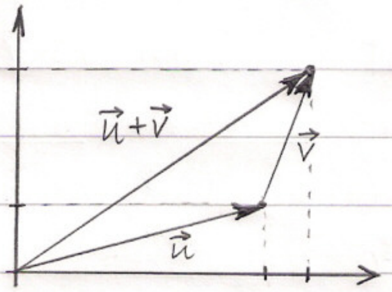


Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$ and let c be a scalar.

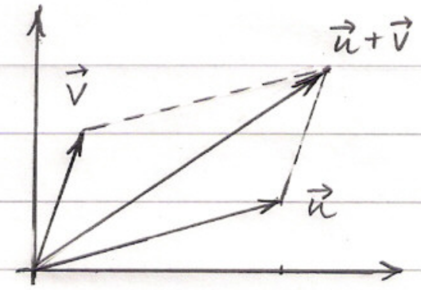
Def. $\vec{u} = \vec{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$.

Def.
- $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$
- $c\vec{u} = (cu_1, cu_2)$

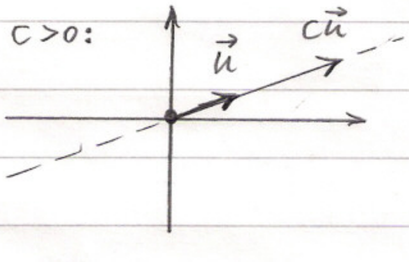
Geometric interpretation of these two operations :



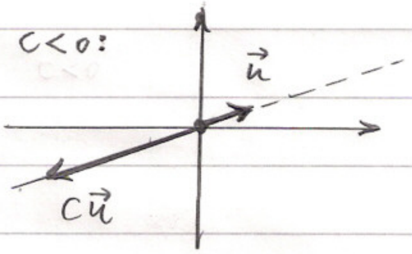
or



$c>0$:



$c<0$:



Theorem: $\|c\vec{u}\| = |c|\cdot\|\vec{u}\|$

Proof: $\|c\vec{u}\| = \|(cu_1, cu_2)\| = \sqrt{(cu_1)^2+(cu_2)^2} = \sqrt{c^2u_1^2+c^2u_2^2}$
$= \sqrt{c^2(u_1^2+u_2^2)} = \sqrt{c^2}\sqrt{u_1^2+u_2^2} = |c|\,\|\vec{u}\|$ q.e.d.

Def. The zero vector $\vec{0} = (0,0)$

The negative of $\vec{u}$ $\quad -\vec{u} = (-1)\vec{u} = (-u_1, -u_2)$

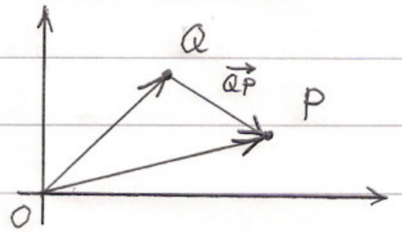
Subtraction $\quad \vec{u}-\vec{v} = \vec{u}+(-\vec{v}) = (u_1-v_1, u_2-v_2)$

Division by a scalar $c \neq 0$ $\quad \frac{\vec{u}}{c} = \frac{1}{c}\vec{u} = \left(\frac{u_1}{c}, \frac{u_2}{c}\right)$

Remark: Two points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$ determine a vector $\overrightarrow{QP}$. One can compute its coordinates as follows:

$$\overrightarrow{QP} = P - Q = (p_1-q_1, p_2-q_2).$$

That is, $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$:



The distance between P and Q is

$$|QP| = \|\overrightarrow{QP}\| = \sqrt{(p_1-q_1)^2+(p_2-q_2)^2}.$$

Ex. For $P=(3,2)$ and $Q=(1,-1)$ we have
$\overrightarrow{QP} = P - Q = (3,2)-(1,-1) = (3-1, 2-(-1)) = (2,3)$.

Theorem: For vectors $\vec{a}, \vec{b}, \vec{c}$ and scalars r, s one has

(1) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(2) $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$

(3) $r(\vec{a}+\vec{b}) = r\vec{a}+r\vec{b}$

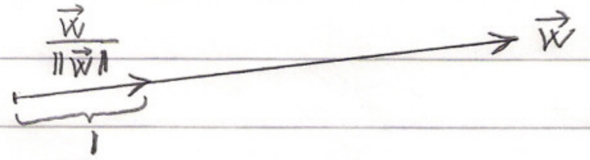
(4) $(r+s)\vec{a} = r\vec{a}+s\vec{a}$

(5) $(rs)\vec{a} = r(s\vec{a})$

(Proof: Write out both sides in coordinates.)

Def. $\vec{u}$ is called a unit vector if $\|\vec{u}\|=1$

Theorem (Normalization)
If $\vec{w} \neq \vec{0}$ then $\frac{\vec{w}}{\|\vec{w}\|}$ is a unit vector pointing in the same direction as $\vec{w}$.



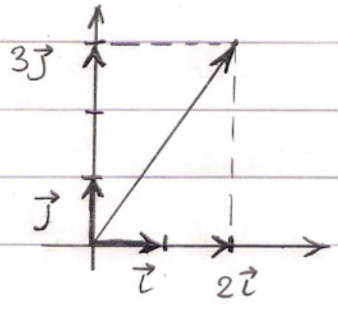
(Proof: $\frac{1}{\|\vec{w}\|} > 0$ and $\left\| \frac{\vec{w}}{\|\vec{w}\|} \right\| = \left\| \frac{1}{\|\vec{w}\|} \vec{w} \right\| = \left| \frac{1}{\|\vec{w}\|} \right| \|\vec{w}\| = \frac{\|\vec{w}\|}{\|\vec{w}\|} = 1$.
q.e.d.)

Def. The standard basis vectors are
$$\vec{i} = (1,0) \quad \text{and} \quad \vec{j} = (0,1).$$

Remark:
$$\vec{u} = (u_1, u_2) = u_1 \vec{i} + u_2 \vec{j}$$

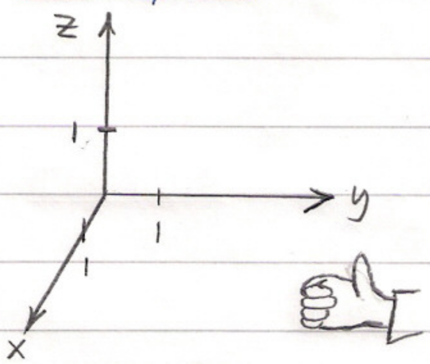
Proof:
$u_1\vec{i} + u_2\vec{j} = u_1(1,0) + u_2(0,1) = (u_1, 0) + (0, u_2) = (u_1, u_2) = \vec{u}$
q.e.d.

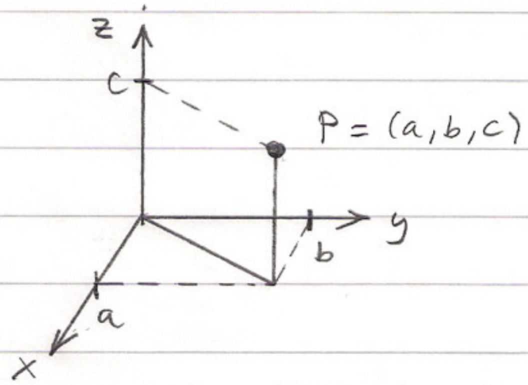
Ex.



$(2,3) = 2\vec{i} + 3\vec{j}$

3D-space





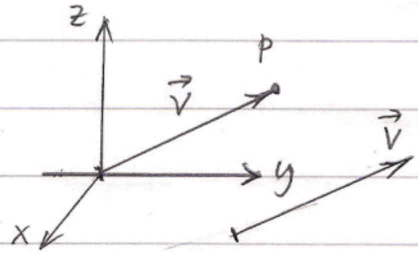
Right-handed coordinate system

Coordinates of a point

<u>Def.</u> A vector in 3D-space is an ordered triple of real numbers

$$\vec{v} = (a, b, c)$$

<u>Geometric interpretation</u>:



For vectors $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and scalar c:

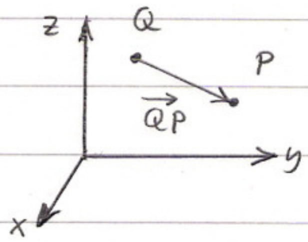
$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ length of $\vec{u}$.

$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$c\vec{u} = (cu_1, cu_2, cu_3)$

As for vectors in the plane one has

- $\|c\vec{u}\| = |c| \, \|\vec{u}\|$
- Two points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ determine a vector $\overrightarrow{QP} = P - Q$.

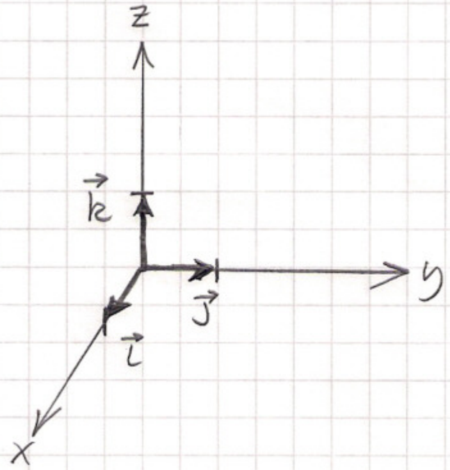


- $|QP| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$

etc.

Standard basis vectors :

$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$



$$\vec{u} = (u_1, u_2, u_3) = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}.$$