7. Session: Areas, Sums and Integrals II

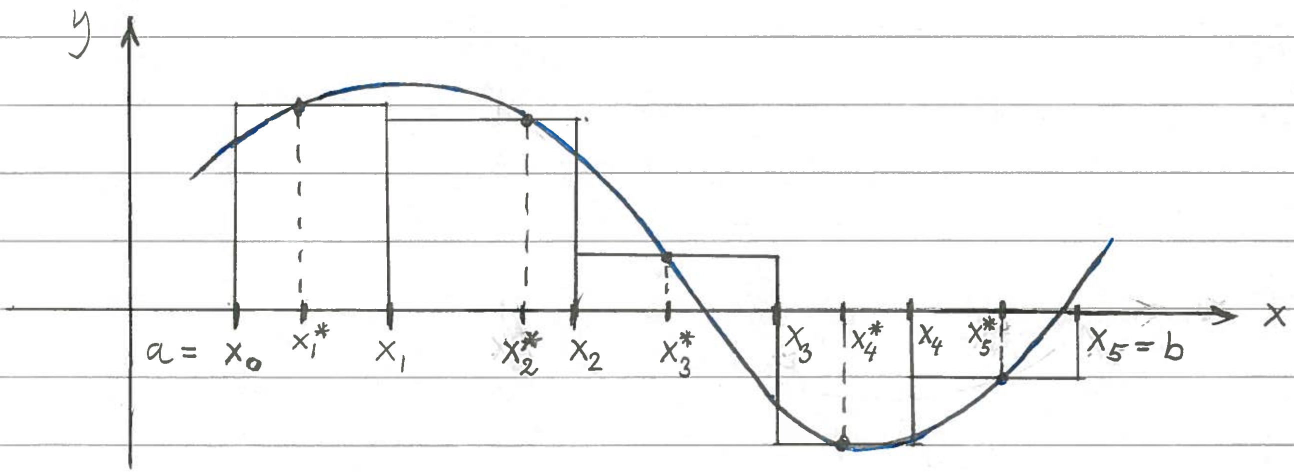
Riemann sums

Let $f(x)$ be a function defined on the interval $[a,b]$.

- A partition $\mathcal{P}$ of $[a,b]$ is an increasing sequence
$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

The length of the subinterval $[x_{i-1}, x_i]$ is $\Delta x_i = x_i - x_{i-1}$.

- A selection $\mathcal{S}$ for the partition $\mathcal{P}$ is a collection of points
$$x_i^* \in [x_{i-1}, x_i] \quad \text{for} \quad i = 1,2,3,\dots,n.$$
- The Riemann sum for $f(x)$ determined by $\mathcal{P}$ and $\mathcal{S}$ is
$$R = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

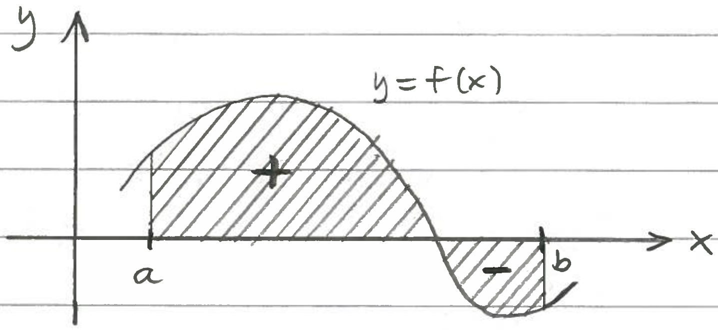


Let $|\mathcal{P}| = \max\{\Delta x_i \mid i = 1,2,\dots,n\}$.

Def. The integral of $f(x)$ from a to b is the number
$$\int_a^b f(x)\,dx = \lim_{|\mathcal{P}|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

provided that this limit exists. If it does, $f(x)$ is said to be integrable on $[a,b]$.

! Note: $\int_a^b f(x)\,dx$ is the area from a to b under the graph $y = f(x)$ and above the x-axis minus the area below the x-axis and above the graph:



Note: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

By definition,

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Theorem: If the function $f(x)$ is continuous on $[a,b]$, then $f(x)$ is integrable on $[a,b]$.

Evaluation of integrals

! Def. An antiderivative of a function $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

Theorem: Let $g(x)$, $x \in]a,b[$ be a function with $g'(x) = 0$ for all x. Then there exists a constant C such that $g(x) = C$ for all $x \in]a,b[$.

Corollary: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then there exists a constant C such that

$$F(x) = G(x) + C.$$

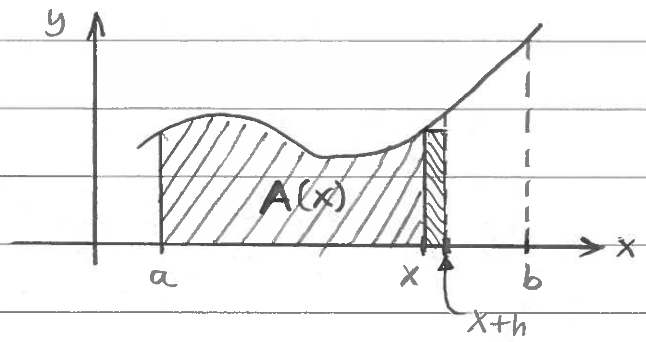
Proof: $(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$ so
$F(x) - G(x) = C \Rightarrow F(x) = G(x) + C$ q.e.d.

Notation: $[F(x)]_a^b = F(b) - F(a)$.

Theorem: If $F(x)$ is an antiderivative of the continuous function $f(x)$ on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Idea of proof:



$$A(x) = \int_a^x f(t)\,dt$$

$$A(a) = 0$$

$$A(b) = \int_a^b f(t)\,dt.$$

For h small, we have:

$$A(x+h) \approx A(x) + f(x)\cdot h \Rightarrow \frac{A(x+h)-A(x)}{h} \approx f(x).$$

It follows that

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x).$$

Thus $A(x)$ is an antiderivative of $f(x)$ such that

$$A(x) = F(x) + C.$$

Finally,

$$\int_a^b f(x)\,dx = A(b) = A(b) - A(a) =$$

$$F(b)+C - (F(a)+C) = \underline{F(b)-F(a)}$$

Ex. $\int_0^2 x^5\,dx = [\frac{1}{6}x^6]_0^2 = \frac{1}{6}\cdot 2^6 - \frac{1}{6}\cdot 0^6 = \frac{64}{6} = \underline{\underline{\frac{32}{3}}}$

Ex. $\int_0^\pi \sin(x)\,dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0))$

$= -(-1) - (-1) = \underline{\underline{2}}$

Ex. $\int_0^1 e^{2x}\,dx = [\frac{1}{2}e^{2x}]_0^1 = \frac{1}{2}e^{2\cdot 1} - \frac{1}{2}e^{2\cdot 0} = \underline{\underline{\frac{1}{2}(e^2-1)}}$

Note: One cannot compute the integral $\int_0^1 e^{x^2}\,dx$ by this method.

Properties of integrals

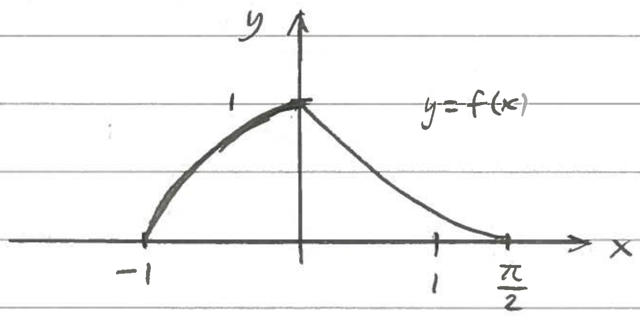
- $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- If $a < c < b$, then
  $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

- If $f(x) \leq g(x)$ for $x \in [a,b]$, then
  $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Ex.
$$f(x) = \begin{cases} 1-x^2, & x \in [-1,0] \\ 1-\sin(x), & x \in [0,\frac{\pi}{2}] \end{cases}$$



$$\int_{-1}^{\frac{\pi}{2}} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{\frac{\pi}{2}} f(x)\,dx$$

$$= \int_{-1}^{0} (1-x^2)\,dx + \int_{0}^{\frac{\pi}{2}} (1-\sin(x))\,dx$$

$$= \left[x - \tfrac{1}{3}x^3\right]_{-1}^{0} + \left[x + \cos(x)\right]_{0}^{\frac{\pi}{2}}$$

$$= 0 - \tfrac{1}{3}\cdot 0^3 - \left(-1 - \tfrac{1}{3}(-1)^3\right) + \tfrac{\pi}{2} + \cos\left(\tfrac{\pi}{2}\right) - \left(0 + \cos(0)\right)$$

$$= \frac{2}{3} + \frac{\pi}{2} - 1 = \underline{\underline{\frac{\pi}{2} - \frac{1}{3}}}$$