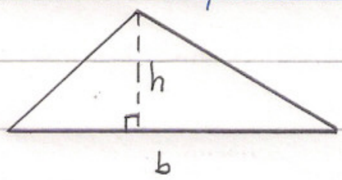
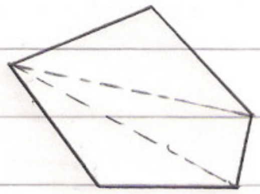
6. Session: Areas, Sums and Integrals I

The consept of area



Area of a triangle:

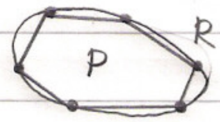
$A = \frac{1}{2} b h$

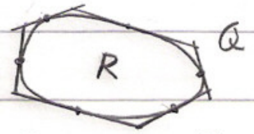


Area of a polygonal figure:

- Divide into nonoverlapping triangles.
- Sum the areas of these triangles.

Area of a curvilinear figure:





P inscribed polygon   Q circumscribed polygon

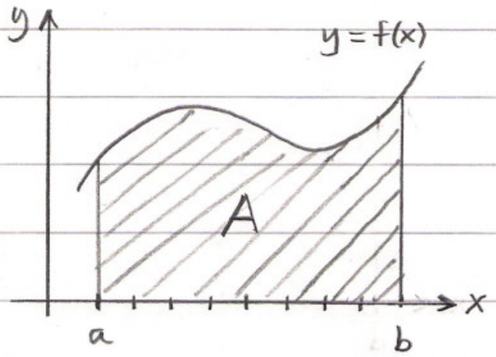
$$a(P) \leq a(R) \leq a(Q)$$

If P and Q have many sides, all short, then $a(P)$ and $a(Q)$ closely approximate $a(R)$.

Area estimate: $$a(R) \approx \frac{a(P)+a(Q)}{2}.$$

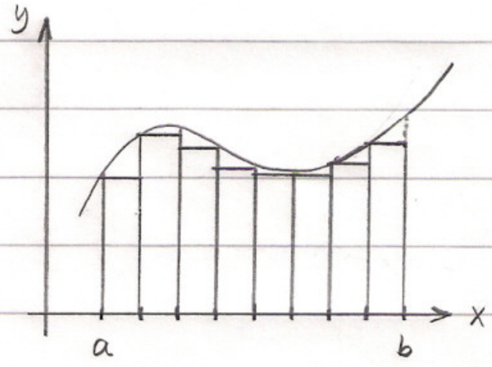
Areas under graphs

Let $f(x)$ be a continuous non-negative function defined on the interval $[a,b]$.

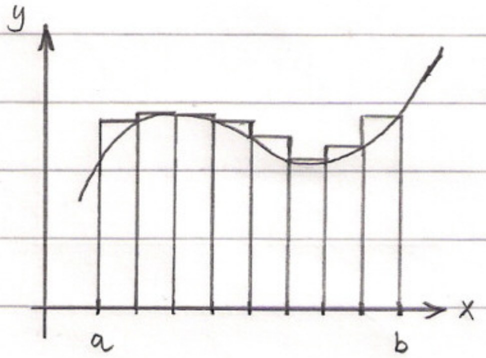


We want to estimate the area A.

Divide $[a,b]$ into n subintervals of equal length.



$\underline{A}_n$ is the sum of the areas of the inscribed rectangles.



$\overline{A}_n$ is the sum of the areas of the circumscribed rectangles

$$\underline{A}_n \leq A \leq \overline{A}_n \qquad A \approx \frac{\underline{A}_n + \overline{A}_n}{2},$$

underestimate overestimate n large

Formulas for $\underline{A}_n$ and $\overline{A}_n$?

Summation notation

Def.: Let $a_1, a_2, \ldots, a_n$ be real numbers. Then,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n .$$

Note: $\displaystyle\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j$

Ex. $\displaystyle\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1+4+9+16+25 = 55$

Ex. $\displaystyle\sum_{i=3}^{7} 2i = 2\cdot 3 + 2\cdot 4 + 2\cdot 5 + 2\cdot 6 + 2\cdot 7 = 50$

Ex. $\displaystyle\sum_{k=1}^{6} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$

Theorem (Rules of summation)

(1) $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

(2) $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

(3) $$\sum_{i=1}^{n} 1 = n$$

Proof:

(1) $$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \dots + ca_n = c(a_1 + a_2 + \dots + a_n) = c\sum_{i=1}^{n} a_i .$$

(2) $$\sum_{i=1}^{n} (a_i+b_i) = a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n = a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i .$$

(3) $$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \dots + 1}_{n} = n$$

q.e.d.

Summation formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{(no proof)}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

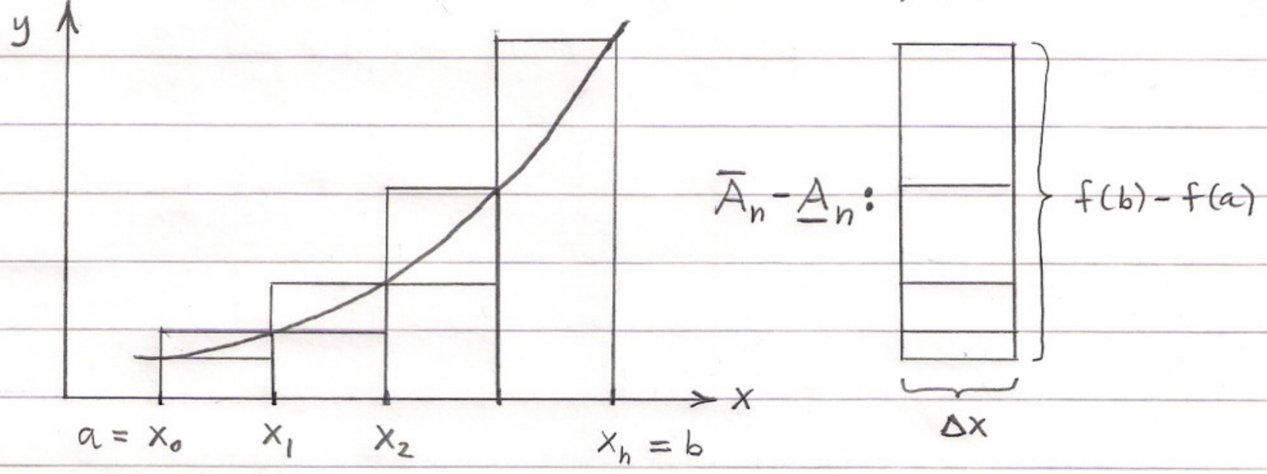
Ex. $$\sum_{i=1}^{100} i = \frac{100 \cdot 101}{2} = 5050$$

Ex. $$\sum_{i=1}^{100} (3i+1) = \sum_{i=1}^{100} 3i + \sum_{i=1}^{100} 1 = 3\sum_{i=1}^{100} i + 100 =$$

$$3 \cdot 5050 + 100 = 15250.$$

Area sums

Let $f(x)$ be a continuous, non-negative and increasing function defined on the interval $[a, b]$.



Divide $[a,b]$ into n subintervals $[x_i, x_{i+1}]$ of the same length $\Delta x = \frac{b-a}{n}$. We have

$$x_i = a + i\,\Delta x \quad \text{for} \quad i = 0, 1, 2, \dots, n.$$

Underestimate : $\underline{A}_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$

Overestimate : $\bar{A}_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$

Note that $\Delta x = \frac{b-a}{n} \to 0$ as $n \to \infty$

Such that

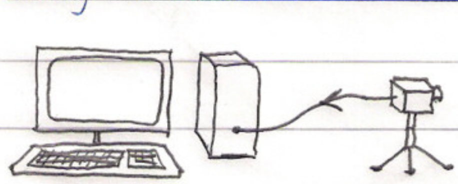
$$\bar{A}_n - \underline{A}_n = \Delta x \cdot (f(b) - f(a)) \to 0 \quad \text{as} \quad n \to \infty.$$

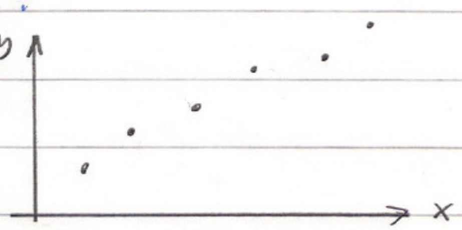
Since $\underline{A}_n \le A \le \bar{A}_n$ for all n, it follows that the area under the graph is

$$A = \lim_{n\to\infty} \underline{A}_n = \lim_{n\to\infty} \bar{A}_n.$$

This result also holds if $f(x)$ is decreasing.

Why are area sums relevant ?





Points on a graph, not $f(x) = 5\sin x + x^2$.