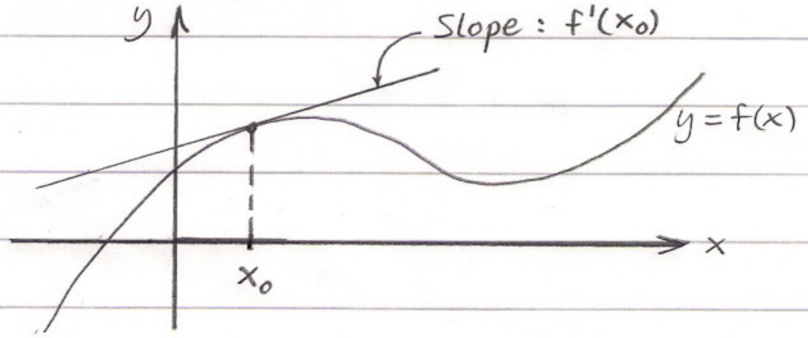
3. Session: Derivatives of Trigonometric Functions

<u>Recall</u>: The derivative $f'(x)$ of a function $f(x)$ is defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Geometric interpretation:



Notation: $f'(x) = \frac{d}{dx} f(x) = D_x f(x).$

<u>Recall:</u> $\frac{d}{dx}(x^n) = n x^{n-1}$,

$\frac{d}{dx}(c) = 0$, c constant.

! <u>Theorem:</u>

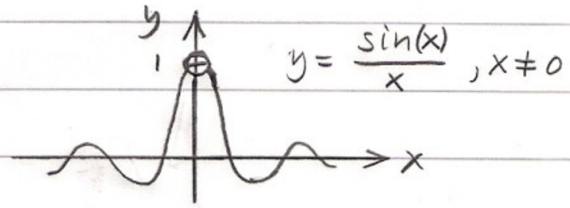
$$\frac{d}{dx} \sin(x) = \cos(x) \qquad (1)$$

$$\frac{d}{dx} \cos(x) = -\sin(x) \qquad (2)$$

<u>Proof of (1)</u>:

One can show that

$$\frac{\sin(\theta)}{\theta} \to 1 \text{ as } \theta \to 0. \qquad (\mathrm{I})$$



It follows that

$$\frac{\cos(\theta) - 1}{\theta} \to 0 \text{ as } \theta \to 0 \quad (\mathrm{II}),$$ since

$$\frac{\cos\theta - 1}{\theta} = \frac{(\cos\theta - 1)\cdot(\cos\theta + 1)}{\theta\cdot(\cos\theta + 1)} = \frac{\cos^2\theta - 1}{\theta\cdot(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta\cdot(\cos\theta + 1)}$$

$$= -\frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\cos\theta + 1} \to -1 \cdot \frac{0}{1+1} = 0 \text{ as } \theta \to 0.$$

Put $f(x) = \sin(x)$. We have

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin(x)}{h} =$$

$$\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} =$$

$$\sin(x)\cdot\frac{\cos(h)-1}{h} + \cos(x)\cdot\frac{\sin(h)}{h} \xrightarrow{(I),(II)}$$

$\sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x)$ as $h \to 0$. q.e.d.

<u>Recall:</u> Differentiation rules

- $(f(x)+g(x))' = f'(x)+g'(x)$
- $(c\,f(x))' = c\,f'(x)$
- $(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x)$.
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$

<u>Theorem:</u>

$$\frac{d}{dx}\tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)} = 1+\tan^2(x). \qquad (1)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x) \qquad (2)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad (3)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad (4)$$

<u>Proof:</u>

(1) $\tan(x) = \frac{\sin(x)}{\cos(x)} \Rightarrow$

$$\frac{d}{dx}\tan(x) = \frac{\frac{d}{dx}(\sin(x))\cdot\cos(x) - \sin(x)\cdot\frac{d}{dx}(\cos(x))}{(\cos(x))^2} =$$

$$\frac{\cos(x)\cdot\cos(x)-\sin(x)\cdot(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} =$$

$\frac{1}{\cos^2(x)} = \sec^2(x) = 1+\tan^2(x)$. OK

(2) - (4) follows by similar calculations. q.e.d.

The Chain Rule

Theorem (The Chain Rule)

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

Ex. $\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot \frac{d}{dx}(x^2) = \cos(x^2) \cdot 2x$
$= 2x \cdot \cos(x^2).$

Ex. $\frac{d}{dx}((3x^2+5)^7) = 7 \cdot (3x^2+5)^6 \cdot \frac{d}{dx}(3x^2+5)$
$= 7 \cdot (3x^2+5)^6 \cdot 6x = 42x \cdot (3x^2+5)^6.$

Ex. $\frac{d}{dx}\left(\frac{1}{2x+1}\right) = \frac{d}{dx}((2x+1)^{-1}) = (-1)(2x+1)^{-2} \cdot 2$
$= -2(2x+1)^{-2} = -\frac{2}{(2x+1)^2}$

! **Remark**: By the chain rule we have

$$\frac{d}{dx}(\sin(kx)) = k \cdot \cos(kx),$$
$$\frac{d}{dx}(\cos(kx)) = -k \cdot \sin(kx),$$

for any constant k.

Ex. $\frac{d}{dt}(\sin(2\pi t)) = 2\pi \cdot \cos(2\pi t)$

Remark: Let $\operatorname{sind}(x)$ and $\operatorname{cosd}(x)$ denote sine and cosine of an angle measured in degrees. We have

$$\operatorname{sind}(x) = \sin\left(\frac{\pi}{180}x\right),$$
$$\operatorname{cosd}(x) = \cos\left(\frac{\pi}{180}x\right).$$

Thus,

$$\frac{d}{dx}(\operatorname{sind}(x)) = \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\right) = \frac{\pi}{180}\operatorname{cosd}(x)$$

and

$$\frac{d}{dx}(\operatorname{cosd}(x)) = -\frac{\pi}{180}\sin\left(\frac{\pi}{180}x\right) = -\frac{\pi}{180}\operatorname{sind}(x).$$