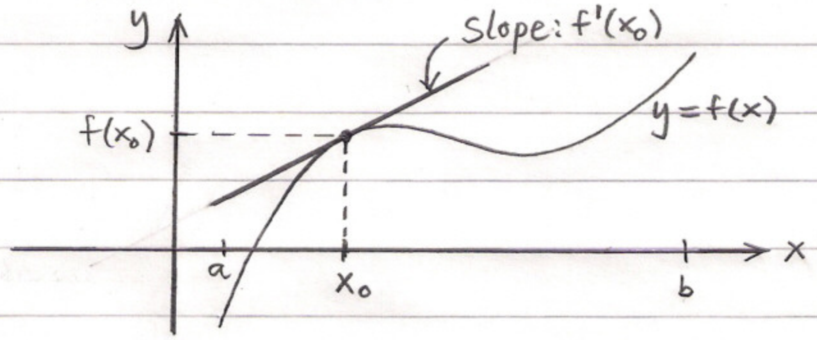
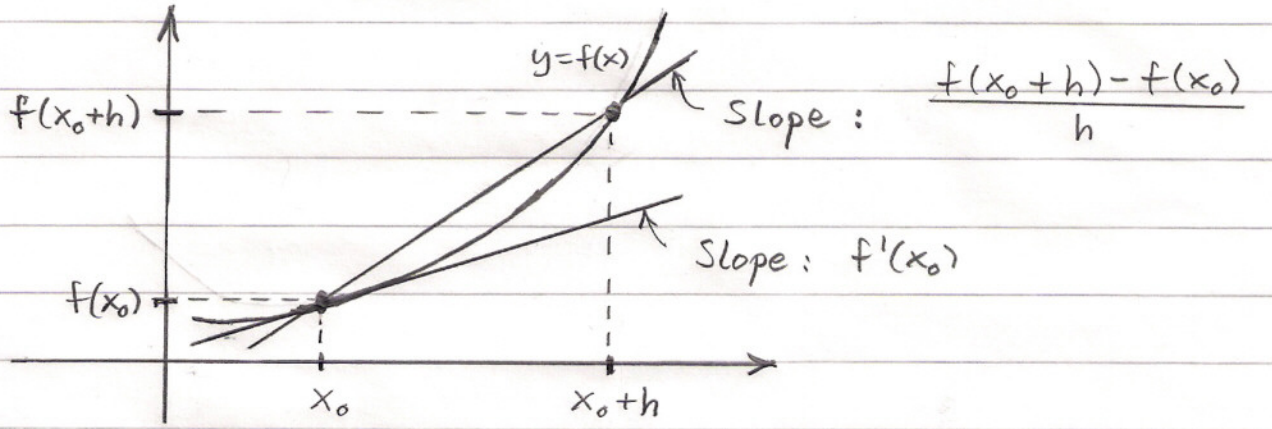
2. Session : The Derivative and Rates of Change

Let $f(x)$ be a function defined on an open interval $I =]a,b[$, and let $x_0 \in I$. Recall that the derivative $f'(x_0)$ of f at x_0 is the slope of the tangent at $(x_0, f(x_0))$ (if it exists).



! <u>Def.</u> $$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$



If $f'(x_0)$ exists for all $x_0 \in I$, we get a new function $f'(x)$, $x \in I$ called <u>the derivative of f.</u>

<u>Ex./Illustration</u> : $f(x) = ax^2+bx+c$, $x \in \mathbb{R}$.

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{a(x_0+h)^2+b(x_0+h)+c-(ax_0^2+bx_0+c)}{h} =$$

$$\frac{a(x_0^2+h^2+2x_0h)+bx_0+bh+c-ax_0^2-bx_0-c}{h} =$$

$$\frac{ah^2+2ax_0h+bh}{h} = ah+2ax_0+b \to 2ax_0+b \quad \text{as } h\to 0$$

Thus, $f'(x_0) = 2ax_0 + b$. Since this holds for any real number x_0, we have that

$$f'(x) = 2ax + b \quad , x \in \mathbb{R}.$$

<u>Notation</u>: $\frac{d}{dx} f(x) = D_x f(x) = f'(x)$.

<u>Remark</u>: The definition of the derivative is rarely used for computations. Instead one uses tabels of derivatives and differentiation rules.

One can prove the following result:

<u>Theorem</u>: For any real number n, one has

$$\frac{d}{dx}(x^n) = n x^{n-1}.$$

<u>Ex.</u>

$$\frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4 \quad ,$$

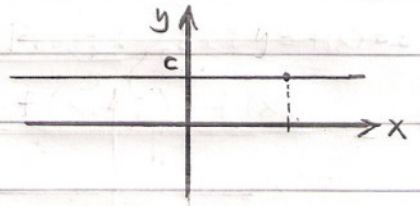
$$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x \quad ,$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1x^{1-1} = 1x^0 = 1 \quad ,$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

<u>Note</u>: For any constant c, one has

$$\frac{d}{dx}(c) = 0.$$



The slope of any tangent to the graph of $f(x) = c$ equals 0.

<u>Differentiation rules</u>:

- $(f(x) + g(x))' = f'(x) + g'(x)$
- $(c f(x))' = c f'(x)$
- $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

- $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

<u>Ex.</u> $\frac{d}{dx}(5x^3 + 2x + 1) = 15x^2 + 2$

<u>Ex.</u> $\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

<u>Rates of change</u>

Let Q be a quantity* that varies with time t,

$$Q = f(t).$$

The change in Q from time t to time $t+\Delta t$ is

$$\Delta Q = f(t+\Delta t) - f(t)$$

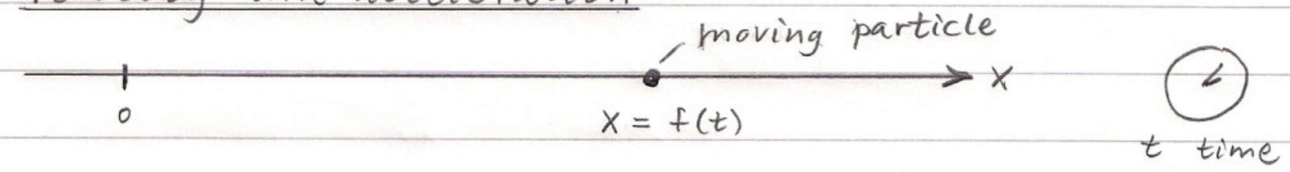
The <u>average rate of change</u> is

$$\frac{\Delta Q}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

The <u>instantaneous rate of change</u> at time t is

$$\lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = f'(t) = \frac{dQ}{dt}.$$

<u>Velocity and acceleration</u>



The change in x from time t to time $t+\Delta t$:

$$\Delta x = f(t+\Delta t) - f(t)$$

Average velocity : $\bar{v} = \frac{\Delta x}{\Delta t}$

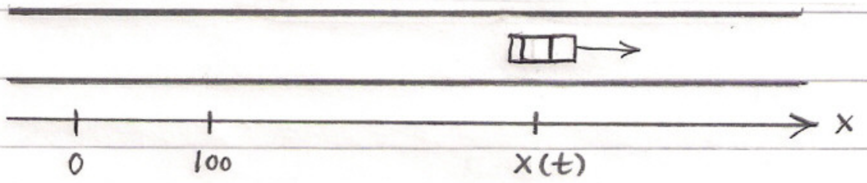
Velocity at time t: $v = \frac{dx}{dt} = f'(t)$

Speed at time t: $|v|$

Acceleration : $a = \frac{dv}{dt} = f''(t)$

* Q could be distance traveled at time t or the size of a population at time t.

Ex.



x in m
t in s

The car starts at time $t=0$. Suppose that

$$x(t) = 2t^2 + 100.$$

The velocity and acceleration at time t is

$$v(t) = x'(t) = 4t$$

$$a(t) = v'(t) = 4$$

We have

$\left.\begin{array}{l} x(0) = 100 \\ v(0) = 0 \end{array}\right\}$: The car starts from rest at time $t=0$ at position 100 m.

$\left.\begin{array}{l} x(10) = 300 \\ v(10) = 40 \end{array}\right\}$: After 10 s the car has traveled 200 m from its initial position and its velocity is 40 m/s

$a(t) = 4$: The acceleration is constantly 4 m/s^2