Det danske eksamenssæt findes ved at vende sættet om

Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

February 17, 2016, 9.00-13.00

This test has 9 pages and 15 problems. In two-sided print. It is allowed to use books, notes, photocopies etc. It is **not** allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.

- Part II is "multiple choice" problems. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each side of your answers. Number each page. Write the total number of pages and the page number on each page of the answers.

NAME: _____

STUDENT NUMBER: _____

COURSE:
- ☐ Aalborg HOLD 1 (Lisbeth Fajstrup)
- ☐ Aalborg HOLD 2 (Jacob Broe)
- ☐ Aalborg HOLD 3 (Nikolaj Hess-Nielsen)
- ☐ Aalborg Robotics (Diego Ruano)
- ☐ AAU-Cph, Dansk hold (Iver Ottosen)
- ☐ AAU-Cph, Engelsk hold (Bedia Møller)

Part I ("Essay-problems")

Problem 1 (10%).

Let
$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 4 & 4 \end{bmatrix}.$$

1. Find a basis for the column space of A.

2. Find a basis for the null space of A.

3. Find a basis for the row space of A.

Problem 2 (10%).

Let
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

1. Find the eigenvalues of A.

2. Find a basis for each of the corresponding eigenspaces.

3. Specify whether A is diagonalisable. If so, find matrices P and D, such that D is diagonal, P is invertible, and $A = PDP^{-1}$.

Part II (Multiple-choice problems)

Problem 3 (5%)

Let R be the row reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Specify the value of R_{13}:

☐ $-\frac{1}{6}$ ☐ -1 ☐ 0 ☐ 1 ☐ 2

Problem 4 (10%).

Consider the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 4 & 4 & -1 & 0 & 0 \\ -2 & 1 & 5 & -2 & 0 \\ 0 & -2 & 1 & 5 & 3 \end{bmatrix}.$$

Mark *all* correct statements below (notice: every incorrect mark *cancels* a correct one).

☐ A is invertible.
☐ The linear transformation $T(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^5$, is surjective (onto).
☐ A is in row-echelon form.
☐ nullity $A = 1$.
☐ rank $A = 4$.
☐ nullity A + rank $A = 5$.
☐ The number -2 is an eigenvalue of A.
☐ A is diagonalizable.
☐ There exists $\mathbf{x} \in \mathbf{R}^5$ with $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$.
☐ $\det A = 24$.

Problem 5 (7%)

Given two 4×4-matrices A and B satisfying $\det A = -3$ and $\det B = 2$. Answer the following questions:

a. The rank of AB is

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

b. The value of $\det(AB^2)$ is

☐ -12 ☐ -6 ☐ 6 ☐ 12 ☐ 14

c. The value of $\det(B^{-1}AB)$ is

☐ -3 ☐ -2 ☐ -1 ☐ 0 ☐ 1

Problem 6 (8%).

Let A and B be $n \times n$ invertible matrices. Let I_n denote the identity matrix of size $n \times n$.

Answer the following 4 true/false questions:

a. We have $\det(AB) = 0$.

☐ True ☐ False

b. We have $((AB)^\top)^{-1} = (A^{-1})^\top (B^{-1})^\top$.

☐ True ☐ False

c. We have $A^{-1}B^{-1}(AB) = I_n$.

☐ True ☐ False

d. We have $B(AB)^{-1} = A^{-1}$.

☐ True ☐ False

Problem 7 (5%)

Let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 4 & 0 \\ -1 & 2 \end{bmatrix}.$$

Let T_A denote the linear transformation $T_A(\mathbf{x}) = A\mathbf{x}$ induced by A, and let T_B and T_{BA} denote the linear transformations induced by B and BA, respectively.

Answer the following 3 true/false questions:

a. We have $T_A : \mathbf{R}^2 \to \mathbf{R}^2$.

☐ True ☐ False

b. We have $T_B : \mathbf{R}^4 \to \mathbf{R}^2$.

☐ True ☐ False

c. We have $T_{BA} : \mathbf{R}^2 \to \mathbf{R}^4$.

☐ True ☐ False

Problem 8 (5%)

The matrix C is given by

$$C = \begin{bmatrix} -3 & 0 & 1 \\ 3 & -2 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$

The determinant of C equals

☐ 0 ☐ -3 ☐ 2 ☐ 5 ☐ -2/3

Problem 9 (5%)

Let

$$A = \begin{bmatrix} 1 & -2 & 7 \\ 1 & -2 & 7 \\ 1 & -2 & 7 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 7 \\ 7 \\ 1 \end{bmatrix}.$$

Answer the following 2 true/false questions:

i. The vector $\mathbf{b}$ is contained in Col A.

☐ True   ☐ False

ii. The vector $\mathbf{b}$ is contained in Null A.

☐ True   ☐ False

Problem 10 (5%)

The following basis is given

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

for $\mathbf{R}^3$. Denote $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Answer the following 3 questions:

i. The vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal.

☐ True   ☐ False

ii. The vectors $\mathbf{b}_2$ and $\mathbf{b}_3$ are orthogonal.

☐ True   ☐ False

ii. $\mathcal{B}$ is an orthonormal basis for $\mathbf{R}^3$.

☐ True   ☐ False

Problem 11 (8%)

The matrix A given by

$$A = \begin{bmatrix} 3 & 1 & 0 & -2 \\ 0 & 3 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ -1 & -3 & 2 & -3 \\ 3 & 0 & -2 & -1 \\ 2 & -2 & 0 & 1 \end{bmatrix}$$

is row-equivalent to the following matrix

$$B = \begin{bmatrix} 3 & 1 & 0 & -2 \\ 0 & 9 & -3 & 9 \\ 0 & 0 & 15 & -9 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer the following 4 questions about A:

a. The rank of A is:

☐ 0   ☐ 1   ☐ 2   ☐ 3   ☐ 4   ☐ 5

b. Given $\mathbf{b} \in \mathbf{R}^6$, the system of equations $A\mathbf{x} = \mathbf{b}$ always has a solution.

☐ True   ☐ False

c. nullity A is:

☐ 0   ☐ 1   ☐ 2   ☐ 3   ☐ 4   ☐ 5

d. The linear transformation $T(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^4$, is injective (one-to-one).

☐ True   ☐ False

Problem 12 (5%)

Consider the system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ -2x_1 - 3x_2 + x_3 = 2 \\ 3x_1 + 5x_2 \qquad = 1 \end{cases}$$

Exactly one of the following statements about the system holds true. Mark the correct statement:

☐ This system has no solution
☐ This system has an infinite number of solutions
☐ This system has a uniquely determined solution
☐ None of the above statements apply.

Problem 13 (5%)

Let $a \in \mathbf{R}$ and consider the matrix

$$A = \begin{bmatrix} 1 & a & a \\ 0 & -a & a \\ 0 & 0 & a \end{bmatrix}.$$

Mark *all* correct statements below (notice: every incorrect mark *cancels* a correct one).

☐ The number a can be chosen such that the columns of A are linearly independent.
☐ The columns of A are linearly dependent regardless of the value of a.
☐ A is invertible for $a \geq 0$.
☐ $\det(A) \leq 0$.

Problem 14 (7%)

The maximal number of linearly independent eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is:

☐ 0   ☐ 1   ☐ 2   ☐ 3   ☐ 4

Problem 15 (5%)

Consider the matrices A and B given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & -2 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & -2 & 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -2 & -2 & 1 \\ 2 & -1 & -2 & -2 \\ 1 & 0 & -2 & 0 \\ 0 & 2 & 1 & -1 \end{bmatrix}.$$

Answer the following two questions:

a. The value of entry $(2,3)$ in AB, i.e. $(AB)_{23}$, is

     ☐ -9   ☐ -5   ☐ 1   ☐ -4   ☐ 2

b. The value of entry $(2,3)$ in BA, i.e. $(BA)_{23}$, is

     ☐ -9   ☐ -5   ☐ 1   ☐ -4   ☐ 2