Answers

Det danske eksamenssæt findes ved at vende sættet om

Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

February 11th, 2015, 9.00-13.00

This test has 7 pages and 12 problems. In two-sided print. It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts. Part I contains “regular problems”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is “multiple choice” problems. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each side of your answers. **Number each page. Write the total number of pages and the page number on each page** of the answers.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE:

- ☐ Aalborg HOLD 1 (v. Lisbeth Fajstrup.)
- ☐ Aalborg HOLD 2 (v. Olav Geil).
- ☐ Aalborg HOLD 3 (v. Leif Kjær Jørgensen).
- ☐ Aalborg HOLD 4 (v. Morten Nielsen).
- ☐ Aalborg HOLD 5 (v. Jacob Broe).
- ☐ Aalborg HOLD 6 (v. Diego Ruano).
- ☐ AAU-Cph, Dansk hold (v. Iver Ottosen).
- ☐ AAU-Cph, English group (w. Bedia A. Møller).
- ☐ Esbjerg, Dansk hold (v. Ulla Tradsborg).
- ☐ Esbjerg, English group (w. Ann-Eva Christensen).

Part I ("regular problems")

Problem 1 (10%).

Let

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 9 & 7 & 14 \end{bmatrix} \quad \text{og} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

1. Find all solutions to the system of equations $Ax = \mathbf{b}$.
2. Find the reduced echelonform for A.

Problem 2 (9%).

Let

$$A = \begin{bmatrix} \sqrt{5} & \frac{\sqrt{2}}{2} & 1 \\ 2\sqrt{5} & 3\sqrt{2} & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \sqrt{5} \\ 2\sqrt{2} \\ 3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \frac{1}{2} & 2 \\ \frac{1}{5} & 1 \\ \frac{3}{10} & 1 \end{bmatrix}$$

Decide and explain for each of the following cases, if the expression makes sen-se. Evaluate the expressions that make sense.

1. AB^T 2. $A\mathbf{c}$ 3. B^TD 4. D^TA

$$\begin{bmatrix} 1 & 0 \\ 2 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 4 \\ 6 & 0 \end{bmatrix}$$

Problem 3 (8%).

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

1. Determine the standardmatrix for T.
2. Determine the standardmatrix for the inverse linear transformation T^{-1}.

Matricen for T $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ for T^{-1} $\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 4 (7%).

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix R is the reduced echelonform of A. (You do not have to show that.)

1. Find a basis for the null space for A. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$
2. Find a basis for the column space for A. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Problem 5 (9%).

Let $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$.

1. Find the eigenvalues of A. 3 and -3
2. Find a basis for each of the associated eigenspaces. For $\lambda = 3$: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $\lambda = -3$: $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$
3. Determine if A is diagonalizable. If so, find matrices P and D, such that D is diagonal, P is invertible (regular) and $A = PDP^{-1}$.

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ OR $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$

Problem 6 (9%).

The subspace W of $\mathbb{R}^4$ has the following basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

1. Determine an orthogonal basis for W using the Gram Schmidt process. $\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$
2. Then determine an orthonormal basis for W. 2) $\left\{ \frac{1}{\sqrt{5}} v_1, v_2, v_3 \right\}$
3. Determine a basis for $W^{\perp}$.

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Problem 7 (8%).

Here we study isometries from $\mathbb{R}^2$ to $\mathbb{R}^2$. The matrices A and B are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and B is the standard matrix for a 90° counter clockwise rotation.

1. Determine B. $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
2. Decide if A is the matrix for a reflection or a rotation. A is a reflection (in $y=x$)
3. Evaluate AB and decide if it is the matrix for a reflection or a rotation.

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ AB is a reflection (in the x-axis)

Problem 8 (10%).

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

In the grading of this, 10 points were given to students who got 1 right and argued that 2,3 would not make sense if $\mathcal{B}$ is not a basis.
10 points were also allocated to students who got 1 right and then solved 2,3 as if $\mathcal{B}$ was a basis

1. Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$. $\mathcal{B}$ is not a basis, since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$
2. A vector $\mathbf{u}$ is given by $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Determine $\mathbf{u}$. $u = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$
3. A linear operator T is given by

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \; T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \; T\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Determine $[T]_{\mathcal{B}}$. $= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

Part II ("multiple choice" Problems)

Problem 9 (6%).

The matrices A, B and C are 3×3.
Moreover, we know that: A is not invertible (not regular), $\det(B) = 2$ and $\det(C) = 3$.
For each of the following three questions, tick off the right answer:

$\det(ABC)$ is

☒ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

$\det(B^3)$ is

☐ 2 ☐ -2 ☐ 3 ☐ 4 ☒ 8 ☐ 16

$\det(C^T B)$ is

☐ $\frac{2}{3}$ ☐ 2 ☐ 3 ☒ 6 ☐ 5 ☐ $\frac{3}{2}$

Problem 10 (8%).

W is a subspace of $\mathbf{R}^4$ and $W^\perp$ is the orthogonal complement.
W has dimension 2. P is the standard matrix for orthogonal projection on W.
Tick off the true statements below. There are three of them.
Notice that if you tick off a wrong statement, it will neutralize a correct tick off. Hence, for example two correct and one incorrect will count as one correct. You cannot get negative points, i.e., one correct and three incorrect count as zero correct.

- ☐ The dimension of $W^\perp$ is 1
- ☒ The dimension of $W^\perp$ is 2
- ☐ P is invertible (regular).
- ☒ If $\mathbf{v}$ is both in W and in $W^\perp$, then $\mathbf{v} = 0$
- ☐ If $\mathbf{v}$ is both in W and in $W^\perp$, then $|\mathbf{v}| = 1$
- ☒ The null space for P is $W^\perp$
- ☐ 2 is an eigenvalue for P
- ☐ $P^2 = I$

Problem 11 (8%).

$$A = \begin{bmatrix} -2 & 1 & 3 & 0 \\ 0 & 3 & 0 & 7 \\ 0 & 0 & 14 & 2 \\ 0 & 0 & 0 & 32 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Tick off the true statements below. There are four of them.
Notice that if you tick off a wrong statement, it will neutralize a correct tick off. Hence, for example two correct and one incorrect will count as one correct. You cannot get negative points, i.e., one correct and three incorrect count as zero correct.

- [ ] A is invertible (regular).
- [ ] A is in reduced row echelonform.
- [x] A is in row echelonform.
- [ ] The null space for A has dimension 1 (Nullity for A is 1)
- [x] The rank of A is 4
- [ ] The equation $Ax = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^5$
- [ ] A is a 4×5 matrix.
- [x] A is a 5×4 matrix.
- [ ] $\text{Nullity}(A) + \text{Rank}(A) = 5$
- [x] $\text{Nullity}(A) + \text{Rank}(A) = 4$

Problem 12 (8%).

The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$$

are column vectors in $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$

Answer the following four true/false problems:

a. $\det A = c$

☐ True ☒ False

b. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent for every choice of a, b, c.

☐ True ☒ False

c. $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent for every choice of a, b, c.

☒ True ☐ False

d. A has at least two different eigenvalues.

☒ True ☐ False

The end of the English version of the exam