Exam in Linear Algebra

First Year at The TEK-NAT Faculty and Health Faculty

February 19th, 2014, 9.00-13.00

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts. Part I contains "regular problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" problems. The answers for Part II must be given on these sheets.

Remember to write your student number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

STUDENT NUMBER: ____________________

COURSE: □AAU-Cph (Bedia Akyar Møller)

□AAU-Esbjerg (Richard Cleyton)

Part I ("regular problems")

Problem 1 (9%).

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

1. Determine the general solution to the equation $A\mathbf{x} = \mathbf{b}$.

Problem 2 (9%).

Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Determine for each of the following 4 situations if the expression makes sense. Evaluate the expressions that make sense. For the expressons that do not make sense, write "the expression is not defined".

1. $A\mathbf{d}$ 2. $(A+B)\mathbf{d}$ 3. BC 4. CB.

Problem 3 (7%).

Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

1. Find the row echelon form of A.
2. Determine the determinant of A.

Problem 4 (8%).

Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

1. Find a basis for W.
2. Argue that $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ belongs to W.
3. Argue that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ does not belong to W.

Problem 5 (10%).

Let $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$.

1. Determine the eigenvalues of A.
2. For each of the eigenvalues determine a basis for the corresponding eigenspace.

Problem 6 (10%).

Consider the subspace W of $\mathbb{R}^4$ with basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

1. Determine an orthogonal basis for W by using the Gram-Schmidt process.

Problem 7 (9%).

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

1. Argue that A is an orthogonal matrix.
2. Argue that B is an orthogonal matrix.
3. Does A correspond to a rotation? (Justify your answer).
4. Does B correspond to a rotation? (Justify your answer).
5. Does AB correspond to a rotation? (Justify your answer).

Problem 8 (8%).

$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

1. Let $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Find B^{-1}.
2. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Determine $[\mathbf{v}]_{\mathcal{B}}$.

Partial answer for
Reexam in Linear Algebra
February 19th, 20014

Problem 1 $\begin{bmatrix}3\\0\\-1\end{bmatrix} + x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}$

Problem 2

1. $\begin{bmatrix}7\\7\end{bmatrix}$

2. $\begin{bmatrix}17\\11\end{bmatrix}$

3. the expression is not defined

4. $\begin{bmatrix}4&1&4\\5&1&5\end{bmatrix}$

Problem 3

1. $\begin{bmatrix}1&0&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ (there are other possible answers)

2. -1

Problem 4

1. $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ (there are other possible answers)

2. $\begin{bmatrix}1\\2\\4\end{bmatrix} = 1\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} + 1\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}$

3. $\left[\begin{array}{ccc|c}1&2&0&0\\2&4&0&1\\3&0&1&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&2&0&0\\0&0&0&1\\3&0&1&0\end{array}\right] \leftarrow$ zero-row followed by non-zero element

Problem 5

1. $\{2,-1\}$

2. for $\lambda = 2$ $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

for $\lambda = -1$ $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$

Problem 6

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ 0 \\ -12 \end{bmatrix} \right\}$$

Problem 7

1. $A^T A = I$
2. $B^T B = I$
3. rotation
4. not a rotation
5. not a rotation

Problem 8

1. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Part II ("multiple choice" problems)

Problem 9 (6%).

Let A be a matrix for which the reduced row echelon form is $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Answer the following three true/false problems:

a. A is invertible (non-singular, regular)

☐ True ☒ False

b. The columns of A are linearly independent.

☐ True ☒ False

c. The rows of A are linearly dependent.

☒ True ☐ False

Problem 10 (8%).

A is a 5×5 matrix with determinant $\det A = 5$.

For each of the following four problems tick off the correct answer:

$\det(A^3) =$

☐ 0 ☐ 5 ☐ 15 ☐ 25 ☒ 125

$\det(A^{-1}) =$

☐ -5 ☐ 0 ☐ 1 ☒ $\frac{1}{5}$ ☐ 5

$\det\left((A^T)^{-1}\right) =$

☐ -5 ☐ 0 ☐ 1 ☒ $\frac{1}{5}$ ☐ 5

The number of vectors in a basis for Col A equals

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☒ 5

Problem 11 (8%).

Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Answer the following four true/false problems:

a. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a generating set for W.

☒ True ☐ False

b. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for W.

☐ True ☒ False

c. $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for W.

☒ True ☐ False

d. $\{\mathbf{u}_1, \mathbf{u}_3\}$ is a generating set for W.

☒ True ☐ False

Problem 12 (8%).

Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

Answer the following three true/false problems:

a. It holds that $T_A : \mathbb{R}^2 \to \mathbb{R}^2$.

☒ True ☐ False

b. It holds that $T_B : \mathbb{R}^2 \to \mathbb{R}^3$.

☐ True ☒ False

c. For all $\mathbf{x}$ in $\mathbb{R}^3$ it holds that $T_A(T_B(\mathbf{x})) = T_{AB}(\mathbf{x})$

☒ True ☐ False