Exam in Linear Algebra

First Year at The TEK-NAT Faculty and Health Faculty

February 20th, 2013, 9.00-13.00

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts. Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets.

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE: ☐AAU-Cph (Iver Ottosen)

☐AAU-Esbjerg (Olav Geil, Torben Tvedebrink, Leif K. Jørgensen)

Part I ("regular exercises")

Exercise 1 (10%).

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

1. Reduce A to its reduced row echelon form.
2. Solve the equation $A\mathbf{x} = \mathbf{b}$ or argue that it does not have any solution.

Exercise 2 (9%).

Let

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Determine for each of the following 4 situations if the expression makes sense. Evaluate the expressions that make sense. For the expressions that do not make sense, write "the expression is not defined".

1. $A\mathbf{c}$ 2. $B\mathbf{c}$ 3. AB 4. $BA\mathbf{c}$.

Exercise 3 (6%).

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

1. Find A^{-1}.
2. Argue that B^{-1} does not exist. (That is, argue that B is not invertible/regular).

Exercise 4 (10%).

Let $A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 3 & 6 & 4 & 4 \\ 0 & -9 & 3 & -6 \end{bmatrix}$.

1. Find a basis for the column space of A.
2. Find a basis for the null space of A.

Exercise 5 (8%).

Let $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$.

1. Find the eigenvalues of A.
2. For each of the eigenvalues determine a basis for the corresponding eigenspace.
3. Decide if A is diagonalizable (justify your answer).

Exercise 6 (8%).

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

1. Reduce A to row echelon form.
2. Calculate the determinant of A.
3. Find the determinant of A^4.

Exercise 7 (9%).

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. The following information are given:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

1. Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
2. Determine the standard matrix for T (that is, determine A such that $T(\mathbf{x}) = A\mathbf{x}$).
3. Is T surjective? (Alternative word is onto).
4. Is T injective? (Alternative word is one-to-one).

Exercise 8 (10%).

The subspace W of $\mathbb{R}^4$ has as basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

1. Determine an orthogonal basis for W by using the Gram-Schmidt process.
2. Then determine an ortonormal basis for W.

Part II ("multiple choice" exercises)

Exercise 9 (6%).

You are given the information that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

Answer the following two true/false exercises:

a. $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible.

☐ True ☐ False

b. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. It holds that $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

☐ True ☐ False

Exercise 10 (6%).

Answer the following three true/false exercises:

a. Let B be an $n \times n$ invertible matrix. Then it holds that $\det B = 0$.

☐ True ☐ False

b. Let B be an $n \times n$ invertible matrix and let $\mathbf{b}$ be an $n \times 1$ column vector. Then it holds that $\mathbf{x} = B^{-1}\mathbf{b}$ is a unique solution to the equation $B\mathbf{x} = \mathbf{b}$.

☐ True ☐ False

c. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let A be the corresponding standard matrix. Then it holds that A is an $m \times n$ matrix.

☐ True ☐ False

Exercise 11 (10%).

Let

$$\begin{bmatrix} 7 & 1 & 1 & 1 & 0 \\ 0 & 9 & 1 & 1 & 1 \\ 0 & 0 & 13 & 1 & 0 \\ 0 & 0 & 0 & 11 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Tick off correct statements among the ten statements below.

(*Only the statements that you have ticked off will contribute to your mark. Among the statements you have ticked off every incorrect statement will neutralize a correct statement. So if you ticked off 5 statements of which 4 are correct and 1 is wrong, you will receive credits for 4-1=3 correct answers. If you ticked of 4 statements of which 2 are correct and 2 are incorrect, you will receive credits for 2-2=0 correct answer (= no credit). You cannot receive a negative number of points. Hence, if you ticked off 4 statements of which 1 is correct and 3 are incorrect you will receive credits for 0 correct answer (= no credit).*)

- ☐ A is quadratic.
- ☐ A is symmetric.
- ☐ A has five different eigenvalues.
- ☐ A is diagonalizable.
- ☐ A is invertible.
- ☐ $\det(A) = 0$.
- ☐ rank $A = 5$.
- ☐ nullity $A = 4$.
- ☐ The columns of A constitute a basis for $\mathbb{R}^5$.
- ☐ The rows of A are linearly dependent.

Exercise 12 (8%).

In this exercise we consider linear operators from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let A be the standard matrix for a counterclockwise rotation with $20°$. Let B be the standard matrix for a counterclockwise rotation with $80°$. Let C be the standard matrix for a reflection about the x-axis.

Answer the following four true/false exercises:

a. $AB = BA$

☐ True ☐ False

b. $CC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

☐ True ☐ False

c. BC is the standard matrix for a reflection about a line.

☐ True ☐ False

d. -1 is an eigenvalue for C.

☐ True ☐ False