Exam in "Linear Algebra"
February 22nd, 2012

First Year at The TEK-NAT Faculty and Health Faculty

It is allowed to use books, notes etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts. Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets.

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

COURSE NUMBER:
- ☐ HOLD 1 (v. Jacob Broe).
- ☐ HOLD 2 (v. Olav Geil).
- ☐ HOLD 3 (v. Leif Kjær Jørgensen).
- ☐ HOLD 4 (v. Bo Rosbjerg).
- ☐ HOLD 5 (v. Martin Raussen).

Del I ("regular exerciser")

Exercise 1 (10%).

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 10 \\ 2 & 4 & 7 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

1. Reduce A to its reduced echelon form.
2. Solve the system of equations $A\mathbf{x} = \mathbf{b}$.

Exercise 2 (6%).

Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Consider the following four expressions. For each expression determine if the expression is defined. If not, write "not defined", otherwise compute the result:

1. AB 2. AB^T 3. $B^T A$ 4. $B\mathbf{c}$.

Exercise 3 (9%).

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

1. Determine A^{-1}.
2. Determine $\left(A^T\right)^{-1}$.

Exercise 4 (9%).

A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \text{og} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

1. Determine $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
2. Write the standard matrix A associated with T.
3. Is A a standard matrix for a rotation? (I.e.: Does T correspond to a rotation?)

Exercise 5 (10%).

1. Show that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   is a basis for $\mathbb{R}^3$.

2. Let

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}.$$

   Determine B^{-1}.

3. A linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ has standard matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

   Determine $[T]_{\mathcal{B}}$.

Exercise 6 (10%).

Given the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. Find the associated eigenvectors.
3. Diagonalise A. I.e. determine a diagonal matrix D and a invertible matrix P such that $A = PDP^{-1}$.

Exercise 7 (8%).

The subspace $W \subseteq \mathbb{R}^4$ has basis

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 7 \\ 0 \end{bmatrix} \right\}.$$

1. Determine an orthogonal basis for W using the Gram-Schmidt procedure.

Exercise 8 (10%).

Given

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}.$$

1. Determine the matrix P_W for the orthogonal projection of $\mathbb{R}^3$ on W.
2. Determine the vectors $\mathbf{w} \in W$ and $\mathbf{z} \in W^\perp$ such that $\mathbf{u} = \mathbf{w} + \mathbf{z}$.

Part II ("multiple choice" exercises)

Exercise 9 (4%).

To different vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ are given such that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Let $H = \text{Span}\{\mathbf{u}, \mathbf{v}\}$.
Exactly one of the following statements are correct. Mark this one:

- ☐ The dimension of H is 2.
- ☐ The dimension of H is 3
- ☐ H can be described as a line in $\mathbb{R}^3$.

Exercise 10 (10%).

Let

$$A = \begin{bmatrix} 4 & 7 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 2 & 10 \end{bmatrix}.$$

Mark *all* the true statements below (note: every wrong answer *eliminates* one correct answer).

- ☐ $\det A = 24$.
- ☐ $\det A = 240$.
- ☐ A is invertible.
- ☐ A is symmetric.
- ☐ A is orthogonal.
- ☐ 4 is an eigenvalue of A.
- ☐ 10 is an eigenvalue of A.
- ☐ $\text{rank } A = 4$.
- ☐ The linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by $T(\mathbf{x}) = A\mathbf{x}$ is injective (one-to-one).
- ☐ The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every vector $\mathbf{b} \in \mathbb{R}^4$.

Exercise 11 (6%).

For a matrix A with 2 rows and 4 columns answer the following three questions.

Determine the maximal rank of A:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

Determine the maximal nullity of A:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

Determine the minimal nullity of A:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

Exercise 12 (8%).

Answer the following four true/false questions:

a. The matrix $A = \begin{bmatrix} 0,6 & -0,8 \\ 0,8 & 0,6 \end{bmatrix}$ is orthogonal.

☐ True ☐ False

b. There exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that is surjective (onto).

☐ True ☐ False

c. Let W be a subspace of $\mathbb{R}^6$ with dimension 5. Then the dimension of $W^\perp$ is 1.

☐ True ☐ False

d. Let W be the subspace from question c. There exists exactly one vector, that belongs to both W and $W^\perp$.

☐ True ☐ False