Exam in “Linear Algebra”
August 18th, 2011

First Year at The TEK-NAT Faculty and Health Faculty

It is allowed to use books, notes etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts. Part I contains “regular exercises”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is “multiple choice” exercises. The answers for Part II must be given on these sheets.

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

NAME: ____

STUDENT NUMBER: ____

Part I ("regular exercises")

Exercise 1 (6%).

Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 1 & 3 \end{bmatrix}.$$

1. Compute $AB + AC$.

Exercise 2 (10%).

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 3 & 1 & 3 \end{bmatrix}.$$

1. Find A^{-1}.
2. Determine the determinant of A.
3. Determine the determinant of A^{-1} and the determinant of A^8.

Exercise 3 (10%).

Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 3 & 4 \\ -3 & -9 & 0 & 3 \end{bmatrix}.$$

1. Find a basis for the column space of A.
2. Find a basis for the null space of A.
3. Find a basis for the row space of A.

Exercise 4 (8%).

Let

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. Find a basis for each of the associated eigenspaces.
3. Diagonalise A. That is, find matrices P and D, such that P is invertible, D is a diagonal matrix and $A = PDP^{-1}$ holds.

Exercise 5 (10%).

Let W be the solution set for

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0. \end{aligned}$$

Let $\mathbf{u} = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$.

1. Determine the orthogonal projection matrix P_W.
2. Find $\mathbf{w} \in W$ and $\mathbf{z} \in W^{\perp}$, such that $\mathbf{u} = \mathbf{w} + \mathbf{z}$.
3. Find the distance from $\mathbf{u}$ to W.

Exercise 6 (8%).

Let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

1. Show that $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}$ is a basis for W.
2. Justify that $\mathbf{v} = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$ is an element of W.
3. Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^3$ and find $[\mathbf{v}]_{\mathcal{B}}$.

Exercise 7 (10%).

We have that

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

can be written as $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{og} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

1. Find the solution of the system of differential equations

$$\begin{aligned} y_1' &= y_1 \qquad\quad + 4y_3 \\ y_2' &= y_1 + 2y_2 - y_3 \\ y_3' &= \qquad\qquad\quad 5y_3 \end{aligned}$$

that satisfies the conditions

$$\begin{cases} y_1(0) &= -1 \\ y_2(0) &= 5 \\ y_3(0) &= 1. \end{cases}$$

Exercise 8 (8%).

This exercise considers linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

1. Write the matrix for a reflection in the second-axis.
2. Write the matrix for a counter-clockwise 45°-rotation about the origin.
3. Write the matrix that corresponds to a reflection in the second-axis (question 8.1) followed by a rotation as specified in question 8.2.

Part II ("multiple choice" exercises)

Exercise 9 (4%).

There are given two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$, that are linearly *dependent*, and $\mathbf{u} \neq \mathbf{v}$. Let

$$H = \text{span}\{\mathbf{u}, \mathbf{v}\}.$$

Mark the true statement below

- ☐ H can be described as a parallelogram in $\mathbf{R}^3$.
- ☐ H can be described as a plane in $\mathbf{R}^3$.
- ☐ H can be described as a line in $\mathbf{R}^3$.

Exercise 10 (10%).

Consider the matrix

$$A = \begin{bmatrix} 2 & 5 & -2 & -1 \\ 0 & -7 & 3 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Mark *all* the true statements below (note: every wrong answer *eliminates* one correct answer).

- ☐ A is not invertible.
- ☐ The linear transformation induced by A is injective (one-to-one).
- ☐ A is on echelon form.
- ☐ nullity $A = 1$.
- ☐ rank $A = 3$.
- ☐ nullity A + rank $A = 3$.
- ☐ The number 2 is a eigenvalue of A.
- ☐ A is on reduced echelon form.
- ☐ There exists a $\mathbf{b} \in \mathbf{R}^4$, such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.
- ☐ $\det A = 0$.
- ☐ $\det A = -14$.

Exercise 11 (6%).

There are given a linear mapping $T\colon \mathbf{R}^8 \to \mathbf{R}^m$. Answer the following two questions.

Determine the minimal value of m, for which T can *not* be surjective (onto):

☐ 0	☐ 2	☐ 4	☐ 6	☐ 8	☐ 10
☐ 1	☐ 3	☐ 5	☐ 7	☐ 9	☐ 11

Determine the minimal value of m, for which T *can* be injective (one-to-one):

☐ 0	☐ 2	☐ 4	☐ 6	☐ 8	☐ 10
☐ 1	☐ 3	☐ 5	☐ 7	☐ 9	☐ 11

Exercise 12 (10%).

Answer the following five true/false questions:

a. A 4×4-matrix with four eigenvalues $-1, 0, 1, 5$ can be diagonalised.

☐ True ☐ False

b. There exists a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

☐ True ☐ False

c. Let W be a subspace of $\mathbf{R}^5$ with dimension 4. Then any set of 4 linear independent of $\mathbf{R}^5$ constitutes a basis for W.

☐ True ☐ False

d. Let A and B be 3×3-matrices. If $AB = O$, then either $A = O$ or $B = O$.

☐ True ☐ False

e. Any symmetric 2×2-matrix can be diagonalised.

☐ True ☐ False