For at finde den danske version af prøven, begynd venligst i den modsatte ende!

Please ignore the Danish version on the back, if you follow the English version of the exam.

Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

24 August 2018, 9:00 – 13:00

This test consists of 9 pages and 14 problems. All problems are "multiple choice" problems. For each problem there is given a number of points.
The total for all 14 problems is 100 points.

It is allowed to use books, notes, xerox copies etc.
It is **not allowed** to use **any electronic devices**.

Your answers must be given by marking the relevant boxes on these sheets.
The evaluation is only based on these markings.

In problems 11, 12, 13 and 14 the evaluation is done following this principle:
Each wrong mark will annul one correct mark.

Remember to fill in your full name together with your student number below.

NAME:______________________________________

STUDENT NUMBER:______________________________________

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Problem 1 (8 points)

1. Is the system of equations

$$\begin{array}{rrrcr} x_1 & -3x_2 & +x_3 & = & 1 \\ 3x_1 & -7x_2 & +7x_3 & = & 5 \\ x_1 & -2x_2 & +3x_3 & = & -4 \end{array}$$

   consistent?

   ☐ Yes ☑ No

2. How many solutions does the system have?

   ☐ 1 ☑ none
   ☐ 3 ☐ infinitely many

3. Is the system of equations

$$\begin{array}{rrrcr} x_1 & -3x_2 & +x_3 & = & 1 \\ 3x_1 & -7x_2 & +7x_3 & = & 5 \\ x_1 & -2x_2 & +3x_3 & = & 2 \end{array}$$

   consistent?

   ☑ Yes ☐ No

4. How many solutions does the system have?

   ☐ 1 ☐ none
   ☐ 3 ☑ infinitely many

Problem 2 (6 points)

There is given an augmented matrix $[A\ \mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & -2 & 4 & 0 \\ 2 & 2 & 2 & -4 & 8 & 2 \\ 2 & 1 & 1 & -2 & 7 & 2 \end{bmatrix}$. Here the reduced echelon form of $[A\ \mathbf{b}]$ is the matrix $[H\ \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

1. Which of the following systems corresponds to the equation $A\mathbf{x} = \mathbf{b}$?

   ☐ $\begin{cases} 3x_1 & & & -2x_4 & +4x_5 & & = & 0 \\ & +6x_2 & & -4x_4 & +8x_5 & +2x_6 & = & 0 \\ & & +4x_3 & -2x_4 & +7x_5 & +2x_6 & = & 0 \end{cases}$

   ☐ $\begin{cases} x_1 & +x_2 & +x_3 & -2x_4 & +4x_5 & & = & 0 \\ 2x_1 & +2x_2 & +2x_3 & -4x_4 & +8x_5 & +2x_6 & = & 0 \\ 2x_1 & +x_2 & +x_3 & -2x_4 & +7x_5 & +2x_6 & = & 0 \end{cases}$

   ☑ $\begin{cases} x_1 & +x_2 & +x_3 & -2x_4 & +4x_5 & = & 0 \\ 2x_1 & +2x_2 & +2x_3 & -4x_4 & +8x_5 & = & 2 \\ 2x_1 & +x_2 & +x_3 & -2x_4 & +7x_5 & = & 2 \end{cases}$

2. What is the rank of the coefficient matrix A?

   ☐ 1 ☑ 2 ☐ 3 ☐ 4 ☐ 5

3. What is the rank of the augmented coefficient matrix $[A\ \mathbf{b}]$?

   ☐ 1 ☐ 2 ☑ 3 ☐ 4 ☐ 5 ☐ 6

4. What is the nullity of the matrix A?

   ☐ 0 ☐ 1 ☐ 2 ☑ 3 ☐ 4 ☐ 5

5. Is it correct that the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ form a basis for Null A ?

   ☐ Yes ☑ No

6. Do the vetors $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for Null A ?

   ☑ Yes ☐ No

Problem 3 (6 points)

A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ has the standard matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 3 \end{bmatrix}$.

1. What is the value of the number n?

   ☐ 2 ☐ 3 ☑ 4 ☐ 5

2. What is the value of the number m?

   ☑ 2 ☐ 3 ☐ 4 ☐ 5

3. What is the rank of A?

   ☑ 2 ☐ 3 ☐ 4 ☐ 5

4. What is the dimension of the null space of A?

   ☐ 0 ☐ 1 ☑ 2 ☐ 3 ☐ 4

5. Is T onto (surjective)?

   ☑ Yes ☐ No

6. Is T one-to-one (injective)?

   ☐ Yes ☑ No

Problem 4 (6 points)

1. Give the value of the determinant of the matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$.

   ☐ 6 ☐ 10 ☐ 11 ☑ -10 ☐ -7

2. Give the value of the determinant of the matrix $B = \begin{bmatrix} -3 & 1 & -2 \\ 5 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

   ☐ 6 ☐ 7 ☑ -7 ☐ -6 ☐ 0

3. Which of the numbers below equals the determinant of the inverse B^{-1}?

   ☐ -3 ☐ $\frac{1}{7}$ ☐ 3

   ☐ $-\frac{1}{6}$ ☑ $-\frac{1}{7}$ ☐ none of them

Problem 5 (10 points)

This problem concerns the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

1. Which of the polynomials below equals A's characteristic polynomial ?

   ☐ $-\lambda^3 + \lambda^2 - 3\lambda + 3$ ☐ $\lambda^3 - 5\lambda^2 + 7\lambda - 3$

   ☑ $-\lambda^3 + 5\lambda^2 - 7\lambda + 3$ ☐ none of them

2. Which of the following vectors are eigenvectors of the matrix A ?

   ☐ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ☑ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3. Is the matrix A diagonalisable ?

   ☑ Yes ☐ No

4. Which of the following numbers is equal to $\det(A)$?

   ☐ 0 ☐ 2 ☑ 3 ☐ 4 ☐ -4 ☐ -3

5. Is the matrix A regular/invertible?

   ☑ Yes ☐ No

Problem 6 (6 points)

Three vectors are given by $\mathbf{u} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \frac{1}{4}\begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$.

1. What is the dimension of the subspace $U = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?

   ☐ 0 ☐ 1 ☐ 2 ☑ 3 ☐ 4

2. Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ an orthonormal set ?

   ☑ Yes ☐ No

3. What is the dimension of $U^\perp$?

   ☐ 4 ☐ 3 ☐ 2 ☑ 1 ☐ 0 ☐ -1 ☐ -2 ☐ -3 ☐ -4

Problem 7 (10 points)

In $\mathcal{R}^3$ there is a plane W given by the equation $x_1 - 2x_2 + 5x_3 = 0$.

1. Mark the matrix that equals the orthogonal projection matrix P_W:

   ☐ $\begin{bmatrix} 29 & 2 & -5 \\ 2 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$ ☐ $\frac{1}{130}\begin{bmatrix} 29 & 2 & -5 \\ 2 & 26 & 10 \\ -5 & 10 & 5 \end{bmatrix}$ ☑ $\frac{1}{30}\begin{bmatrix} 29 & 2 & -5 \\ 2 & 26 & 10 \\ -5 & 10 & 5 \end{bmatrix}$

2. Mark the orthogonal projection $P_W\mathbf{v}$ of the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 15 \\ 0 \end{bmatrix}$ onto W:

   ☐ $\begin{bmatrix} 30 \\ 15 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 30 \\ 390 \\ 150 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 13 \\ 5 \end{bmatrix}$ ☐ $\begin{bmatrix} \frac{2}{5} \\ 0 \\ \frac{13}{5} \end{bmatrix}$

Problem 8 (8 points)

A rotation $T\colon \mathcal{R}^2 \to \mathcal{R}^2$ in the plane, counter-clockwise around Origo with the angle of rotation $\theta = \frac{\pi}{3}$, has the standard matrix A.

1. Which of the following matrices is equal to A?

   ☐ $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ ☑ $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

2. Is the matrix A regular/invertible?

   ☑ Yes ☐ No

3. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ constitute an ordered basis $\mathcal{B}$ for $\mathcal{R}^2$. Which of the following matrices corresponds to the matrix $[T]_{\mathcal{B}}$ that describes the rotation T with respect to the ordered basis $\mathcal{B}$?

   ☐ $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ☑ $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

4. Is the matrix A diagonalisable?

   ☐ Yes ☑ No

Problem 9 (8 points)

There is given a matrix M and a vector $\mathbf{v}$ as

$$M = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

1. The vector $\mathbf{v}$ is an eigenvector for M. What is the corresponding eigenvalue ?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☑ 4

2. The number $\lambda = 1$ is an eigenvalue of M. What is the dimension of the corresponding eigenspace ?

   ☐ 0 ☑ 1 ☐ 2 ☐ 3 ☐ 4

3. Give the determinant of M:

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☑ 4

4. Is M diagonalisable ?

   ☐ Yes ☑ No

Problem 10 (6 points)

In this problem A, B are 5×5-matrices fulfilling $\det A = 2$ and $\det(AB) = -8$.

1. What is the value of $\det(2A)$?

   ☐ 32 ☐ -4 ☐ -64 ☐ -8 ☑ 64 ☐ -32

2. What is the value of $\det B^3$?

   ☐ 32 ☐ -4 ☑ -64 ☐ -8 ☐ 64 ☐ -32

3. What is the value of $\det(B^{-1}A^T)$?

   ☐ -6 ☐ $-\frac{1}{8}$ ☐ $\frac{3}{2}$ ☑ $-\frac{1}{2}$

Problem 11 (6 point, with annulment)

For the matrices $A = \begin{bmatrix} 5 & 2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & -2 & 5 \end{bmatrix}$ one may calculate their products $C = AB$ og $D = BA$.

Mark the true ones among the following statements about the entries c_{ij} in C, respectively d_{ij} in D:

- ☑ $c_{11} = d_{11}$
- ☐ $c_{12} = d_{12}$
- ☐ $c_{13} = d_{13}$
- ☑ $c_{22} = d_{22}$
- ☐ $c_{23} = d_{23}$
- ☑ $c_{33} = d_{33}$

Problem 12 (6 points, with annulment)

This concerns the matrix $B = \begin{bmatrix} 2 & 2 & -1 \\ 3 & -1 & 5 \\ 0 & 2 & -1 \end{bmatrix}$ and the vectors $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$, both in $\mathcal{R}^3$.

Mark the correct statements in the list below:

- ☑ $\mathbf{b}$ is contained in the column space Col B.
- ☑ $\mathbf{c}$ is contained in the column space Col B.
- ☐ $\mathbf{b}$ is contained in the null space Null B.
- ☐ $\mathbf{c}$ is contained in the null space Null B.
- ☑ The column space Col B is equal to $\mathcal{R}^3$.
- ☐ The null space Null B is equal to $\mathcal{R}^3$.

Problem 13 (8 points, with annulment)

The problem concerns the vectors, all in $\mathcal{R}^3$,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1. Which of the following sets of vectors are linearly independent?

   - ☑ $\mathbf{v}_3, \mathbf{v}_4$
   - ☐ $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_7$
   - ☑ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$
   - ☑ $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$
   - ☐ $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$
   - ☑ $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$

2. Which of the following sets of vectors span $\mathcal{R}^3$?

   - ☐ $\mathbf{v}_3, \mathbf{v}_4$
   - ☐ $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_7$
   - ☑ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$
   - ☑ $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$
   - ☑ $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$
   - ☑ $\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$

Problem 14 (6 points, with annulment)

The following commands are entered into MATLAB's Command Window:

```
>> a = [1; 0; 3; 4];
>> b = [2; 2; 6; 2];
>> c = [1; 1; 4; 1];
>> d = [1; 0; 3; 4];
>> e = [4; 4; 12; 4];
>> f = [2; 2; 1; 3];
>> C = [a b c d e f];
>> rref(C);

ans =

 1   0   0   1   0    1
 0   1   0   0   2    3
 0   0   1   0   0   -5
 0   0   0   0   0   -2
```

Mark among the following statements the correct ones:

☐ `c` is a row vector.

☐ `C` is a 6×4 matrix.

☑ `C` is a 4×6 matrix.

☑ One calculates the nullity of C (dimension of Null C) by entering
`>> 6 - rank (C);`

☐ One calculates the nullity of C (dimension of Null C) by entering
`>> 5 - rank (C);`

☑ The dimension of C's column space is equal to 4.