Reexam in Linear Algebra

First Year at the Faculty of Engineering and Science
and the Technical Faculty of IT and Design

21 February 2018

The present exam set consists of 9 numbered pages with 14 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use book, notes etc. It **is not** allowed to use electronic devices.

your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Problem 1 (5 points)

A system of equations is given by

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 7 \\ 2x_1 + 7x_3 &= 15 \\ x_1 - 3x_2 - x_3 &= 8. \end{aligned}$$

Mark the correct statement below.

☐ The system of equations has no solutions.

☐ The unique solution of the system is $x_1 = 3$, $x_2 = 0$, $x_3 = 2$.

☐ There are infinitely many solutions. One of these is $x_1 = 3$, $x_2 = 0$, $x_3 = 2$.

☐ The solution set of the system is a subspace of $\mathcal{R}^3$.

Problem 2 (6 points)

Let W be the subspace of $\mathcal{R}^4$ with basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}.$$

The Gram-Schmidt process applied to the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ produces the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, which constitute an orthogonal basis for W. Here $\mathbf{v}_1 = \mathbf{u}_1$. What is the vector $\mathbf{v}_2$?

☐ $\begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -1 \\ 4 \\ -4 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Problem 3 (7 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 5 & 3 \\ 1 & 1 & -2 & 1 \\ 9 & 8 & 0 & 3 \end{bmatrix}.$$

From these one forms the matrices $C = AB$ and $D = A^{-1}B$.

(a) (2 points). What is the size of matrix C?

☐ 4×3 ☐ 3×3 ☐ 4×4 ☐ 3×4 ☐ 3×7

(b) (2 points). What is the entry c_{13}?

☐ 1 ☐ -1 ☐ 8 ☐ 2 ☐ 4

(c) (3 points). What is the entry d_{13}?

☐ 0 ☐ 1 ☐ -2 ☐ 7 ☐ 9

Problem 4 (8 points)

Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -1 \end{bmatrix}.$$

(a) (3 points). What is the dimension of W?

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 12

(b) (5 points). Which one of the following vectors lies in W?

☐ $\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix}$

Problem 5 (6 points)

A matrix is given by

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 3 & 0 & 2 & 0 \\ 5 & 1 & 3 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}.$$

(a) (4 points). What is the determinant of A?

☐ -3 ☐ 1 ☐ 0 ☐ 3 ☐ 5 ☐ 11

(b) (2 points). Is A an invertible matrix?

☐ Yes ☐ No

Problem 6 (10 points)

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be the basis for $\mathcal{R}^2$ with $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Furthermore, let $T : \mathcal{R}^2 \to \mathcal{R}^2$ be a linear transformation. The standard matrix for T is

$$A = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}.$$

(a) (7 points). What is the matrix representation $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$?

☐ $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ ☐ $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

(b) (3 points). What are the eigenvalues of T?

☐ -1 and 1 ☐ -2 and 1
☐ 2 and 5 ☐ There are none

Problem 7 (10 points)

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 7 \\ 1 & -2 & 4 \\ 3 & 0 & 6 \end{bmatrix}.$$

(a) (2 points). What is the value of n?

☐ 1 ☐ 2 ☐ 3 ☐ 4

(b) (2 points). What is the value of m?

☐ 1 ☐ 2 ☐ 3 ☐ 4

(c) (2 points). Is T injective (one-to-one)?

☐ Yes ☐ No

(d) (2 points). Is T surjective (onto)?

☐ Yes ☐ No

(e) (2 points). Does there exist a non-zero vector $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{0}$?

☐ Yes ☐ No

Problem 8 (5 points)

Consider the matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

What are the eigenvalues?

☐ -1 and 3
☐ 1 and -1
☐ 2 and 3
☐ 3 with multiplicity 2
☐ -2 and 5
☐ There are none

Problem 9 (10 points)

A matrix is given by

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}.$$

Answer the questions below. (*Remark: Each wrong mark cancels a true mark in question (a) and (b).*)

(a) (3 points). Which of the following vectors are eigenvectors of A?

☐ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} -8 \\ 10 \\ 7 \end{bmatrix}$

(b) (3 points). Which of the following numbers are eigenvalues of A?

☐ -2 ☐ -1 ☐ 0 ☐ 2 ☐ 4

(c) (2 points). Is A diagonalizable?

☐ Yes ☐ No

(d) (2 points). Is A invertible?

☐ Yes ☐ No

Problem 10 (6 points)

Let A, B and C be 5×5-matrices with $\det(A) = -1$, $\det(B) = 2$ and $\det(C) = 7$.

(a) (2 points). What is $\det(A^5)$?

☐ -5 ☐ 5 ☐ 0 ☐ -1 ☐ 1 ☐ 25

(b) (2 points). What is $\det(A^T BC)$?

☐ 8 ☐ -14 ☐ 10 ☐ 9 ☐ 0 ☐ -9

(c) (2 points). What is $\det(3A)$?

☐ -3 ☐ -9 ☐ 9 ☐ -15 ☐ 22 ☐ -243

Problem 11 (12 points)

Three vectors are given by

$$\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 7 \\ 3 \end{bmatrix}$$

and a subspace is defined as $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) (2 points). What is the dot product $\mathbf{v}_1 \bullet \mathbf{v}_2$?

☐ -2 ☐ 0 ☐ 3 ☐ 5 ☐ -8

(b) (2 points). What is the dimension of W?

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

(c) (2 points). What is the dimension of $W^\perp$?

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

(d) (3 points). What is the orthogonal projection of $\mathbf{u}$ on W?

☐ $\begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 1 \\ 4 \\ -3 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 6 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 0 \\ 6 \\ 4 \\ 4 \end{bmatrix}$

(e) (3 points). What is the orthogonal projection of $\mathbf{u}$ on $W^\perp$?

☐ $\begin{bmatrix} 2 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 4 \\ -3 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 3 \\ 2 \\ -4 \\ 1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 5 \\ 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}$

Problem 12 (7 points)

A matrix is defined by

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

(a) (2 points). What is the dimension of the null space $\text{Null}(A)$?

□ 0 □ 1 □ 2 □ 3 □ 4 □ 8

(b) (5 points). Which one of the following sets is a basis for $\text{Null}(A)$?

□ $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

□ $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 3 \\ 3 \end{bmatrix} \right\}$

□ $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$

□ $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$

□ $\left\{ \begin{bmatrix} -2 \\ 4 \\ 3 \\ 0 \end{bmatrix} \right\}$

□ $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 6 \end{bmatrix} \right\}$

Problem 13 (4 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Which of the following sets are linearly independent? (*Remark: Each wrong mark cancels a true mark in this problem.*)

□ $\{\mathbf{v}_2\}$

□ $\{\mathbf{v}_5\}$

□ $\{\mathbf{v}_1, \mathbf{v}_3\}$

□ $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

□ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

□ $\{\mathbf{v}_1, \mathbf{v}_5\}$

□ $\{\mathbf{v}_2, \mathbf{v}_4\}$

□ $\{\mathbf{v}_3, \mathbf{v}_4\}$

Problem 14 (4 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 9 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \\ 8 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 2 \\ 9 \\ 6 \\ 10 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 7 \\ 4 \\ 4 \\ -21 \\ 24 \end{bmatrix}.$$

One would like to write $\mathbf{w}$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. The MATLAB Command Window is used as follows:

```
>> A = [1 2 3 2 7; 2 0 3 9 4; 3 -1 1 6 4; -1 9 0 10 -21; 5 7 8 3 24];
>> rref(A)

ans =

  1   0   0   0   2
  0   1   0   0  -1
  0   0   1   0   3
  0   0   0   1  -1
  0   0   0   0   0
```

How can $\mathbf{w}$ be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$?

☐ $\mathbf{w} = -\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 + 2\mathbf{v}_4$

☐ $\mathbf{w} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4$

☐ $\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 - \mathbf{v}_4$

☐ $\mathbf{w}$ can't be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$.