

Reexam in Linear Algebra

First Year at the Faculty of Engineering and Science
and the Technical Faculty of IT and Design

21 February 2018

The present exam set consists of 9 numbered pages with 14 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use book, notes etc. It **is not** allowed to use electronic devices.

your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below

Good luck!

NAME: _____

STUDENT NUMBER: _____

Answers

Problem 1 (5 points)

A system of equations is given by

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 7 \\ 2x_1 + 7x_3 &= 15 \\ x_1 - 3x_2 - x_3 &= 8.\end{aligned}$$

Mark the correct statement below.

- The system of equations has no solutions.
- The unique solution of the system is $x_1 = 3, x_2 = 0, x_3 = 2$.
- There are infinitely many solutions. One of these is $x_1 = 3, x_2 = 0, x_3 = 2$.
- The solution set of the system is a subspace of \mathcal{R}^3 .

Problem 2 (6 points)

Let W be the subspace of \mathcal{R}^4 with basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}.$$

The Gram-Schmidt process applied to the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ produces the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, which constitute an orthogonal basis for W . Here $\mathbf{v}_1 = \mathbf{u}_1$. What is the vector \mathbf{v}_2 ?

- $\begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ -1 \\ 4 \\ -4 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Problem 3 (7 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 5 & 3 \\ 1 & 1 & -2 & 1 \\ 9 & 8 & 0 & 3 \end{bmatrix}.$$

From these one forms the matrices $C = AB$ and $D = A^{-1}B$.

(a) (2 points). What is the size of matrix C ?

- 4×3 3×3 4×4 3×4 3×7

(b) (2 points). What is the entry c_{13} ?

- 1 -1 8 2 4

(c) (3 points). What is the entry d_{13} ?

- 0 1 -2 7 9

Problem 4 (8 points)

Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ 4 \\ -1 \end{bmatrix}.$$

(a) (3 points). What is the dimension of W ?

- 1 2 3 4 12

(b) (5 points). Which one of the following vectors lies in W ?

- $\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix}$

Problem 5 (6 points)

A matrix is given by

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 3 & 0 & 2 & 0 \\ 5 & 1 & 3 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}.$$

(a) (4 points). What is the determinant of A ?

- -3 1 0 3 5 11

(b) (2 points). Is A an invertible matrix?

- Yes No

Problem 6 (10 points)

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be the basis for \mathcal{R}^2 with $\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Furthermore, let $T : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ be a linear transformation. The standard matrix for T is

$$A = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}.$$

(a) (7 points). What is the matrix representation $[T]_{\mathcal{B}}$ of T with respect to the basis \mathcal{B} ?

- $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

(b) (3 points). What are the eigenvalues of T ?

- -1 and 1 -2 and 1
 2 and 5 There are none

Problem 7 (10 points)

Let $T : \mathcal{R}^n \rightarrow \mathcal{R}^m$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -2 & 7 \\ 1 & -2 & 4 \\ 3 & 0 & 6 \end{bmatrix}.$$

(a) (2 points). What is the value of n ?

- 1 2 3 4

(b) (2 points). What is the value of m ?

- 1 2 3 4

(c) (2 points). Is T injective (one-to-one)?

- Yes No

(d) (2 points). Is T surjective (onto)?

- Yes No

(e) (2 points). Does there exist a non-zero vector \mathbf{u} such that $T(\mathbf{u}) = \mathbf{0}$?

- Yes No

Problem 8 (5 points)

Consider the matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

What are the eigenvalues?

- 1 and 3 1 and -1
 2 and 3 3 with multiplicity 2
 -2 and 5 There are none

Problem 9 (10 points)

A matrix is given by

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}.$$

Answer the questions below. (Remark: Each wrong mark cancels a true mark in question (a) and (b).)

(a) (3 points). Which of the following vectors are eigenvectors of A ?

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -8 \\ 10 \\ 7 \end{bmatrix}$

(b) (3 points). Which of the following numbers are eigenvalues of A ?

-2 -1 0 2 4

(c) (2 points). Is A diagonalizable?

Yes No

(d) (2 points). Is A invertible?

Yes No

Problem 10 (6 points)

Let A , B and C be 5×5 -matrices with $\det(A) = -1$, $\det(B) = 2$ and $\det(C) = 7$.

(a) (2 points). What is $\det(A^5)$?

-5 5 0 -1 1 25

(b) (2 points). What is $\det(A^T B C)$?

8 -14 10 9 0 -9

(c) (2 points). What is $\det(3A)$?

-3 -9 9 -15 22 -243

Problem 11 (12 points)

Three vectors are given by

$$\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{3} \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 7 \\ 3 \end{bmatrix}$$

and a subspace is defined as $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) (2 points). What is the dot product $\mathbf{v}_1 \bullet \mathbf{v}_2$?

- 2 0 3 5 -8

(b) (2 points). What is the dimension of W ?

- 1 2 3 4 5

(c) (2 points). What is the dimension of W^\perp ?

- 1 2 3 4 5

(d) (3 points). What is the orthogonal projection of \mathbf{u} on W ?

- $\begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ 4 \\ -3 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 6 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 0 \\ 6 \\ 4 \\ 4 \end{bmatrix}$

(e) (3 points). What is the orthogonal projection of \mathbf{u} on W^\perp ?

- $\begin{bmatrix} 2 \\ 2 \\ -2 \\ 3 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 4 \\ -3 \\ 2 \\ 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 2 \\ -4 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}$

Problem 12 (7 points)

A matrix is defined by

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$$

(a) (2 points). What is the dimension of the null space $\text{Null}(A)$?

- 0 1 2 3 4 8

(b) (5 points). Which one of the following sets is a basis for $\text{Null}(A)$?

- $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}$ $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 3 \\ 3 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ $\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$
- $\left\{ \begin{bmatrix} -2 \\ 4 \\ 3 \\ 0 \end{bmatrix} \right\}$ $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 6 \end{bmatrix} \right\}$

Problem 13 (4 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Which of the following sets are linearly independent? (Remark: Each wrong mark cancels a true mark in this problem.)

- $\{\mathbf{v}_2\}$ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
 $\{\mathbf{v}_5\}$ $\{\mathbf{v}_1, \mathbf{v}_5\}$
 $\{\mathbf{v}_1, \mathbf{v}_3\}$ $\{\mathbf{v}_2, \mathbf{v}_4\}$
 $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ $\{\mathbf{v}_3, \mathbf{v}_4\}$

Problem 14 (4 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 9 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \\ 8 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 2 \\ 9 \\ 6 \\ 10 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 7 \\ 4 \\ 4 \\ -21 \\ 24 \end{bmatrix}.$$

One would like to write \mathbf{w} as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.
The MATLAB Command Window is used as follows:

```
>> A = [1 2 3 2 7; 2 0 3 9 4; 3 -1 1 6 4; -1 9 0 10 -21; 5 7 8 3 24];  
>> rref(A)
```

```
ans =
```

```
1 0 0 0 2  
0 1 0 0 -1  
0 0 1 0 3  
0 0 0 1 -1  
0 0 0 0 0
```

How can \mathbf{w} be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

$\mathbf{w} = -\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 + 2\mathbf{v}_4$

$\mathbf{w} = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4$

$\mathbf{w} = 2\mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3 - \mathbf{v}_4$

\mathbf{w} can't be written as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.