Exam in Linear Algebra

First Year at the Faculty of Engineering and Science
and the Technical Faculty of IT and Design

5 Januar 2018

The present exam set consists of 9 numbered pages with 15 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100.

It is allowed to use book, notes etc. It **is not** allowed to use electronic devices.

your answers must be marked directly in this exam set. The grading of the exam is based solely on these marks. Remember to write your **full name** and **student number** below

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

Answers

Problem 1 (6 points)

A system of equations is given by

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ 2x_1 - 4x_2 + 3x_3 &= 7 \\ -x_1 + 2x_2 &= 1. \end{aligned}$$

Mark the correct statement below.

☐ The system of equations has no solutions.

☐ The only solution of the system is $x_1 = 1$, $x_2 = 1$, $x_3 = 3$.

☑ There are infinitely many solutions of the system. One of these is $x_1 = -1$, $x_2 = 0$, $x_3 = 3$.

☐ The system of equations has precisely two solutions.

Problem 2 (7 points)

Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}.$$

(a) (4 points). Which one of the following vectors does not belong to W?

☐ $\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) (3 points). What is the dimension of W?

☐ 1 ☑ 2 ☐ 3 ☐ 6 ☐ 9

Problem 3 (5 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & 7 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

By matrix multiplications one gets the matrix $C = AB$.

(a) (2 points). What is the size of matrix C?

☑ 3×2 ☐ 2×3 ☐ 3×3 ☐ 3×5 ☐ 5×3

(b) (3 points). What is the entry c_{21}?

☐ -1 ☐ 0 ☐ 3 ☑ 6 ☐ 5

Problem 4 (5 points)

Consider the following six matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 7 \\ -1 & 5 \\ 1 & 1 \end{bmatrix}.$$

(a) (2 points). For which value of i is A_i *not* an elementary matrix?

☐ 1 ☑ 2 ☐ 3 ☐ 4

(b) (3 points). For which value of i is $A_i B = C$?

☐ 1 ☐ 2 ☐ 3 ☑ 4

Problem 5 (4 points)

A matrix is given by

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

What is the determinant of the matrix?

☑ -4 ☐ 3 ☐ -2 ☐ 0 ☐ 8 ☐ -6

Problem 6 (10 points)

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 2 & 4 & 2 \end{bmatrix}.$$

(a) (2 points). What is the value of n?

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☑ 5

(b) (2 points). What is the value of m?

☐ 1 ☐ 2 ☑ 3 ☐ 4 ☐ 5

(c) (2 points). Is T injective (one-to-one)?

☐ Yes ☑ No

(d) (2 points). Is T surjective (onto)?

☐ Yes ☑ No

(e) (2 points). Do there exist two different vectors $\mathbf{u}$ and $\mathbf{v}$ such that $T(\mathbf{u}) = T(\mathbf{v})$?

☑ Yes ☐ No

Problem 7 (5 points)

Consider the matrix

$$\begin{bmatrix} -5 & 4 \\ -8 & 7 \end{bmatrix}.$$

What are its eigenvalues?

- ☐ -2 and 5
- ☐ 2 and -5
- ☑ -1 and 3
- ☐ 1 and -3
- ☐ -2 with multiplicity 2
- ☐ there are none

Problem 8 (10 points)

A matrix and a vector are given by

$$A = \begin{bmatrix} -1 & -6 & 8 \\ -4 & -3 & 8 \\ -4 & -6 & 11 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (2 points). The vector $\mathbf{v}$ is an eigenvector of A. What is the corresponding eigenvalue?

☐ -2 ☐ 0 ☑ 1 ☐ 4

(b) (3 points). The number $\lambda = 3$ is an eigenvalue of A. What is the dimension of the corresponding eigenspace?

☐ 0 ☐ 1 ☑ 2 ☐ 3

(c) (3 points). Is A diagonalizable?

☑ Yes ☐ No

(d) (2 points). What is the determinant of A?

☐ -3 ☐ 10 ☑ 9 ☐ -33

Problem 9 (10 points)

Three vectors are given by

$$\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 5 \end{bmatrix},$$

and a subspace is defined as $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) (2 points). Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ an orthonormal set?

☑ Yes   ☐ No

(b) (2 points). What is the dimension of W?

☐ 1   ☑ 2   ☐ 3   ☐ 4

(c) (2 points). What is the dimension of $W^{\perp}$?

☐ 1   ☑ 2   ☐ 3   ☐ 4

(d) (2 points). What is the orthogonal projection of $\mathbf{u}$ on W?

☐ $\begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(e) (2 points). What is the distance from $\mathbf{u}$ to W?

☐ $\sqrt{2}$ ☐ 2   ☐ $2\sqrt{3}$ ☑ 4

Problem 10 (4 points)

Let

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & a & 0 \\ 1 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Mark the combination which makes Q an orthogonal matrix.

☐ $a = 1, b = -1, c = 1$

☐ $a = -1, b = 1, c = 2$

☐ $a = 1, b = 2, c = \sqrt{2}$

☑ $a = -1, b = 1, c = \sqrt{2}$

☐ $a = 2, b = -1, c = 1$

☐ $a = 1, b = -1, c = 0$

Problem 11 (7 points)

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be the basis for $\mathcal{R}^2$ with $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Furthermore, let $T : \mathcal{R}^2 \to \mathcal{R}^2$ be a linear transformation. The matrix representation of T with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

What is the standard matrix of T?

- [x] $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$
- [ ] $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$
- [ ] $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
- [ ] $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Remark. The questions in Problem 12 are evaluated using the following principle: Each wrong mark cancels a true mark.

Problem 12 (8 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

(a) (4 points). Which of the following sets are linearly independent?

- [x] $\{\mathbf{v}_1, \mathbf{v}_2\}$
- [x] $\{\mathbf{v}_2, \mathbf{v}_3\}$
- [x] $\{\mathbf{v}_5\}$
- [x] $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
- [ ] $\{\mathbf{v}_3, \mathbf{v}_5, \mathbf{v}_6\}$
- [ ] $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$

(b) (4 points). Which of the following sets span $\mathcal{R}^4$?

- [ ] $\{\mathbf{v}_1, \mathbf{v}_2\}$
- [x] $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
- [ ] $\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$
- [ ] $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
- [ ] $\{\mathbf{v}_1, \mathbf{v}_5, \mathbf{v}_6\}$
- [x] $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$

Problem 13 (6 points)

Let A and B be 4×4-matrices with $\det(A) = -2$ and $\det(B) = 5$.

(a) (2 points). What is $\det(A^3)$?

☐ -6 ☐ 6 ☑ -8 ☐ 8 ☐ -16 ☐ 16

(b) (2 points). What is $\det(A^T B^{-1})$?

☐ $-\frac{5}{2}$ ☐ -10 ☐ 10 ☐ $\frac{1}{5}$ ☑ $-\frac{2}{5}$ ☐ -50

(c) (2 points). What is $\det(2B)$?

☐ 10 ☐ 20 ☑ 80 ☐ $\frac{5}{2}$ ☐ -10 ☐ 50

Problem 14 (7 points)

A subspace of $\mathcal{R}^4$ is given as $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ -4 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 19 \\ -2 \\ 2 \\ 26 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 9 \end{bmatrix}.$$

One wants to determine a basis for W and use the MATLAB Command Window as follows:

```
>> A = [ 2 4 1 1 19 2; -1 -2 1 4 -2 3; 1 2 -1 -4 2 1; 3 6 1 0 26 9];
>> rref(A)

ans =

 1   2   0   -1   7   0
 0   0   1    3   5   0
 0   0   0    0   0   1
 0   0   0    0   0   0
```

Which one of the sets below is a basis for W?

☐ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
☑ $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$
☐ $\{\mathbf{v}_5, \mathbf{v}_6\}$
☐ $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
☐ $\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$
☐ $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$

Problem 15 (6 points)

The linear transformation

$$T : \mathcal{R}^2 \to \mathcal{R}^2; \quad T\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{13} \begin{bmatrix} -12 & 5 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

describes a reflection about a line through the origin. Which one of the following vectors lies on the line of reflection?

☑ $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ☐ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$