Det danske eksamenssæt findes ved at vende sættet om

Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

January 8th, 2016, 9.00-13.00

This test has 9 pages and 15 problems. In two-sided print. It is allowed to use books, notes, photocopies etc. It is **not** allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is "multiple choice" problems. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each side of your answers. Number each page. Write the total number of pages and the page number on each page of the answers.

NAME: ________________________________

STUDENT NUMBER: ________________________________

COURSE:

- ☐ Aalborg HOLD 1 (Lisbeth Fajstrup)
- ☐ Aalborg HOLD 2 (Jacob Broe)
- ☐ Aalborg HOLD 3 (Nikolaj Hess-Nielsen)
- ☐ Aalborg Robotics (Diego Ruano)
- ☐ AAU-Cph, Dansk hold (Iver Ottosen)
- ☐ AAU-Cph, Engelsk hold (Bedia Møller)

Part I ("Essay-problems")

Problem 1 (10%).

Let

$$A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ -4 & 2 & -2 & -2 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

1. Find a basis for the column space of A.
2. Find a basis for the null space of A.
3. Find a basis for the row space of A.

Problem 2 (10%).

Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

1. Find A^{-1}.
2. Calculate the determinant of A.
3. Calculate the determinant of A^{-1}.

Part II (Multiple-choice problems)

Problem 3 (5%)

Let R be the row reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

Specify the value of R_{14}:

☐ $-\frac{1}{6}$ ☐ $\frac{1}{3}$ ☐ $\frac{3}{8}$ ☐ 1 ☐ $-\frac{1}{8}$

Problem 4 (10%).

A 4×4-matrix A is given with $\det(A) = 2$. The matrix B is given by

$$B = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 7 & 2 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Mark all correct statements below (notice: every incorrect mark cancels a correct one).

☐ $\det(A^{-1}) = -\frac{1}{2}$.
☐ $\det(B) = -6$.
☐ $\det(AB) = \det(BA)$.
☐ B is invertible.
☐ $\det(A^2B^2) = 36$.
☐ The number 3 is an eigenvalue of B.
☐ B is in reduced echelon-form.
☐ There exists $\mathbf{b} \in \mathbf{R}^4$, such that the linear system of equations $A\mathbf{x} = \mathbf{b}$ *fails* to have a solution.
☐ $\det(-B) = -3$.
☐ $\det(AB) = -12$.

Problem 5 (7%)

Let

$$W = \text{Span}\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} \text{ and } A = \begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\2 & 3 & 0\end{bmatrix}.$$

Answer the following 3 questions about W and A.

a. The dimension of W equals

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

b. The dimension of $W^{\perp}$ equals

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

c. The rank of A is

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

Problem 6 (8%).

Answer the following 5 true/false questions:

a. Let W be af subspace of $\mathbf{R}^4$ with dimension 3. Three linearly independent vectors in W automatically form a basis for W.

☐ True ☐ False

b. Let Q be a 3×3 matrix with $\det(Q) = 1$. Then Q is an orthogonal matrix.

☐ True ☐ False

c. The columns in a 4×4 orthogonal matrix form a basis for $\mathbf{R}^4$.

☐ True ☐ False

d. Let A be a 4×4 matrix with eigenvalues $1, 2, 0$ og -3. Then A can be diagonalized.

☐ True ☐ False

e. Let A be a 4×4 matrix with eigenvalues $1, 2, 0$ og -3. Then A is invertible.

☐ True ☐ False

Problem 7 (5%)

Which of the following statements are true (notice: every incorrect mark cancels a correct one):

- ☐ Four linearly independent vectors in $\mathbf{R}^4$ form a basis for $\mathbf{R}^4$.
- ☐ Every subspace of $\mathbf{R}^n$ has a basis.
- ☐ Every set of vectors in $\mathbf{R}^n$ can be expanded to a basis for $\mathbf{R}^n$
- ☐ Let W be a non-trivial subspace of $\mathbf{R}^n$. It is *not* always possible to find an orthonormal basis for W.

Problem 8 (5%)

Let C be given by

$$C = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & -1 & 1 \end{bmatrix}.$$

Then $\det(C)$ is

☐ 2/3 ☐ -3 ☐ 3 ☐ 6 ☐ -2

Problem 9 (5%)

Let

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix} \quad \text{og} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}.$$

Answer the following 2 true/false questions:

i. The vector $\mathbf{b}$ is contained in $\text{Col}(A)$.

☐ True ☐ False

ii. The vector $\mathbf{b}$ is contained in $\text{Nul}(A)$.

☐ True ☐ False

Problem 10 (5%)

The following basis is given

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

for $\mathbf{R}^3$. Denote $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and consider the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Answer the following two questions:

i. $\mathcal{B}$ is an orthogonal basis $\mathbf{R}^3$.

☐ True ☐ False

ii. The second coordinate of $[\mathbf{v}]_{\mathcal{B}}$ is given by:

☐ $-\sqrt{2}$ ☐ -3 ☐ 1 ☐ 2 ☐ $\frac{1}{\sqrt{2}}$

Problem 11 (8%)

The matrix A given by

$$A = \begin{bmatrix} -1 & -1 & 3 & -3 & -3 & -1 \\ 2 & 2 & -1 & 1 & -2 & 3 \\ -3 & -3 & 1 & 1 & 0 & -3 \\ -3 & -3 & 2 & -1 & -3 & -1 \end{bmatrix}$$

is row-equivalent to the following matrix

$$B = \begin{bmatrix} -1 & -1 & 3 & -3 & -3 & -1 \\ 0 & 0 & -5 & 5 & 8 & -1 \\ 0 & 0 & 0 & -10 & 19 & -8 \\ 0 & 0 & 0 & 0 & 33 & -26 \end{bmatrix}.$$

Answer the following 4 questions about A:

a. The rank of A is:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

b. Given $\mathbf{b} \in \mathbf{R}^4$, the system of equations $A\mathbf{x} = \mathbf{b}$ always has a solution.

☐ True ☐ False

c. nullity(A) is:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

d. The linear transformation $T : \mathbf{R}^6 \to \mathbf{R}^4$, given by $T(\mathbf{x}) = A\mathbf{x}$, is injective (one-to-one).

☐ True ☐ False

Problem 12 (5%)

Consider the system of equations

$$\begin{cases} -x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 - x_3 = 2 \\ -x_1 - x_2 \phantom{{}- x_3} = 0 \end{cases}$$

Exactly one of the following statements about the system holds true. Mark the correct statement:

- ☐ This system has no solution
- ☐ This system has an infinite number of solutions
- ☐ This system has a uniquely determined solution
- ☐ None of the above statements apply.

Problem 13 (5%)

Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Exactly one of the following statements holds true. Mark the correct statement:

- ☐ The columns of A are linearly independent
- ☐ $\det(A) = 1$
- ☐ A is not invertible
- ☐ None of the above statements apply.

Problem 14 (7%)

The maximal number of linearly independent eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

is:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

Problem 15 (5%)

Consider the matrix product AB, where

$$A = \begin{bmatrix} -3 & 0 & 2 & -2 & 2 & 3 \\ -3 & 3 & 1 & 3 & -3 & 3 \\ -2 & -2 & -3 & 1 & 1 & -3 \\ -2 & 1 & 1 & 2 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 1 & 3 \\ -2 & 2 & 3 & -3 \\ -2 & -2 & -1 & 1 \\ 1 & -3 & -2 & -1 \\ 1 & 1 & -3 & -3 \\ -1 & 2 & 1 & 1 \end{bmatrix}.$$

Answer the following two questions:

a. The value of entry $(1,1)$ in AB, i.e. $(AB)_{11}$, is

☐ -12 ☐ -9 ☐ 1 ☐ -13 ☐ 12

b. The value of entry $(3,2)$ in AB, i.e. $(AB)_{32}$, is

☐ -10 ☐ -8 ☐ 0 ☐ $\frac{11}{3}$ ☐ 19