Det danske eksamenssæt findes ved at vende sættet om

Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

January 6th, 2015, 9.00-13.00

This test has 8 pages and 12 problems. In two-sided print. It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts. Part I contains "regular problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" problems. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each side of your answers. **Number each page. Write the total number of pages and the page number on each page** of the answers.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE:

- ☐ Aalborg HOLD 1 (v. Lisbeth Fajstrup.
- ☐ Aalborg HOLD 2 (v. Olav Geil).
- ☐ Aalborg HOLD 3 (v. Leif Kjær Jørgensen).
- ☐ Aalborg HOLD 4 (v. Morten Nielsen).
- ☐ Aalborg HOLD 5 (v. Jacob Broe).
- ☐ Aalborg HOLD 6 (v. Diego Ruano).
- ☐ AAU-Cph, Dansk hold (v. Iver Ottosen).
- ☐ AAU-Cph, English group (w. Bedia A. Møller).
- ☐ Esbjerg, Dansk hold (v. Ulla Tradsborg).
- ☐ Esbjerg, English group (w. Ann-Eva Christensen).

Part I (”regular problems”)

Problem 1 (10%).

Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 5 & 3 \end{bmatrix} \quad \text{og} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

1. Find all solutions to the system of equations $Ax = \mathbf{b}$.

Problem 2 (9%).

Let

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \sqrt{3} \\ 3\sqrt{2} & 2\sqrt{3} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ \frac{2}{3} & \frac{1}{2} & 2 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 3 \\ \frac{2}{3} & 0 \\ \frac{1}{6} & 1 \end{bmatrix} \quad \text{og} \quad \mathbf{d} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix}$$

Decide and explain for each of the following cases, if the expression makes sen-se. Evaluate the expressions that make sense.

1. AB 2. AC 3. BC 4. $A\mathbf{d}$

Problem 3 (7%).

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

1. Determine A^{-1}.
2. Determine $(A^T)^{-1}$.

Problem 4 (7%).

Let $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 3 & 4 & 0 \\ 2 & 6 & 0 & -2 \end{bmatrix}$

1. Find a basis for the Null space of A.
2. Find a basis for the column space of A.

Problem 5 (10%).

Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

1. Find the eigenvalues of A.
2. Find a basis for each of the associated eigenspaces.
3. Determine if A is diagonizable. If so, find matrices P og D, such that D is diagonal, P is invertible (regular) and $A = PDP^{-1}$.

Problem 6 (10%).

The subspace W of $\mathbb{R}^4$ has the following basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

1. Show that the vector $\mathbf{w} = \begin{bmatrix} 0 \\ 4 \\ -3 \\ 0 \end{bmatrix}$ is in $W^{\perp}$.
2. Find a basis for $W^{\perp}$.

Problem 7 (8%).

Let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

1. Determine an orthogonal basis for W using the Gram Schmidt process.
2. Then determine an orthonormal basis for W.

Problem 8 (9%).

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map whose matrix wrt. $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

1. Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.
2. Let $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. Determine $\mathbf{u}$.
3. Determine $[T(\mathbf{u})]_{\mathcal{B}}$ and $T(\mathbf{u})$.

Part II ("multiple choice" Problems)

Problem 9 (6%).

The matrices A and B are 3×3 and $\det A = 2$ and $\det B = -1$. For each of the following three questions, tick off the right answer:

$\det(AB^T)$ is

☐ -3 ☐ -2 ☐ -1 ☐ 1 ☐ 2 ☐ 3

$\det(2A)$ is

☐ -4 ☐ -2 ☐ 2 ☐ 4 ☐ 8 ☐ 16

$\det(A^{-1}BA)$ is

☐ $-\frac{1}{2}$ ☐ -2 ☐ -1 ☐ -4 ☐ 2 ☐ $\frac{1}{2}$

Problem 10 (8%).

Lad

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix R is the reduced echelon form of a certain matrix A

For each of the following three questions, tick off the right answer:

The rank of A is

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

☐ cannot be determined from R.

The dimension of the null space (the nullity) of A is

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

☐ cannot be determined from R.

The dimension of the row space of A is

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

☐ cannot be determined from R.

Problem 11 (8%).

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are in $\mathbf{R}^3$.
The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
A is the matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

Tick off the true statements below. There are five of them.
Notice that if you tick off a wrong statement, it will neutralize a correct tick off. Hence, for example two correct and one incorrect will count as one correct. You cannot get negative points, i.e., one correct and three incorrect count as zero correct. :

- ☐ Span($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$) has dimension 4.
- ☐ Span($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$) has dimension 3.
- ☐ It is not possible to determine the dimension of Span($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$)
- ☐ $\mathbf{v}_4$ is in Span($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$).
- ☐ A is a 3×4 matrix.
- ☐ A is a 4×3 matrix.
- ☐ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbf{R}^3$.
- ☐ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is a basis for $\mathbf{R}^3$.
- ☐ The equation $Ax = \mathbf{b}$ has at least one solution.
- ☐ The equation $Ax = \mathbf{b}$ has at most one solution.
- ☐ The equation $Ax = \mathbf{b}$ has exactly one solution.

Problem 12 (8%).

In the following we study linear operators from $\mathbb{R}^2$ to $\mathbb{R}^2$.

The matrices A and B are both 2×2. A is the standard matrix for a reflection in a line in $\mathbb{R}^2$ and B is the standard matrix for a rotation by an angle θ, where $0° < \theta < 90°$.

Answer the following five true/false problems:

a. $\det A = 1$

☐ True ☐ False

b. AB represents a rotation.

☐ True ☐ False

c. B has two different eigenvalues.

☐ True ☐ False

d. A has two different eigenvalues.

☐ True ☐ False

e. $A^{-1} = A$.

☐ True ☐ False

The end of the English version of the exam