# Exam in Linear Algebra

### First Year at The TEK-NAT Faculty and Health Faculty

### January 7th, 2014, 9.00-13.00

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts. Part I contains "regular problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" problems. The answers for Part II must be given on these sheets.

Remember to write your full name (including middle names) together with your student number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

NAME: ______________________________

STUDENT NUMBER: ______________________________

COURSE: ☐AAU-Cph (Bedia Akyar Møller)

☐AAU-Esbjerg (Richard Cleyton)

## Part I ("regular problems")

### Problem 1 (10%).

Consider the following system of linear equations:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 4x_2 & + & 5x_3 & = & 1. \end{array}$$

1. Determine the corresponding coefficient matrix.
2. Determine the corresponding augmented matrix.
3. Solve the system.

### Problem 2 (9%).

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

Determine for each of the following 4 situations if the expression makes sense. Evaluate the expressions that make sense. For the expressons that do not make sense, write "the expression is not defined".

1. $AB$ 2. $BA$ 3. $BC$ 4. $A(B+C)$.

### Problem 3 (7%).

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

1. Determine $A^{-1}$.
2. Find the determinant of $A$.

## Problem 4 (7%).

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$.

1. Find a basis for the column space of $A$.
2. Find a basis for the row space of $A$.

## Problem 5 (10%).

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

1. Determine the eigenvalues of $A$.
2. For each of the eigenvalues determine a basis for the corresponding eigenspace.

## Problem 6 (10%).

Consider the subspace $W$ of $\mathbb{R}^4$ with basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

1. Determine an orthogonal basis for $W$ by using the Gram-Schmidt process.
2. Then determine an orthonormal basis for $W$.

## Problem 7 (8%).

Let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

1. Argue that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an orthonormal basis for $W$.

2. Determine $\mathbf{w}$ and $\mathbf{z}$ such that $\mathbf{w}$ belongs to $W$, $\mathbf{z}$ belongs to $W^{\perp}$, and $\mathbf{u} = \mathbf{w} + \mathbf{z}$.

## Problem 8 (9%).

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. You are given the information that

$$T\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{og} \quad T\left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

1. Find $T\left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$.

2. Find $T\left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

3. Determine the standard matrix for $T$ (that is, determine $A$ such that $T(\mathbf{x}) = A\mathbf{x}$).

## Part II ("multiple choice" problems)

### Problem 9 (6%).

Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

For each of the following three questions tick off the correct answer:

The dimension of $W$ equals

☐ 0 ☐ 1 ☒ 2 ☐ 3

The dimension of $W^{\perp}$ equals

☐ 0 ☒ 1 ☐ 2 ☐ 3

nullity $A$ equals

☐ 0 ☒ 1 ☐ 2 ☐ 3

**Problem 10 (8%).**

$A$ is a $4 \times 4$ matrix with determinant $\det A = -3$.

For each of the following four questions tick off the correct answer:

$\det(A^2) =$

☐ $-9$ ☐ $-6$ ☐ $0$ ☐ $6$ ☒ $9$

$\det(A^{-1}) =$

☐ $-1$ ☒ $-\frac{1}{3}$ ☐ $0$ ☐ $\frac{1}{3}$ ☐ $1$

$\det(A^T) =$

☒ $-3$ ☐ $-1$ ☐ $0$ ☐ $1$ ☐ $3$

$\det(A^T A) =$

☐ $-9$ ☐ $1$ ☐ $0$ ☐ $3$ ☒ $9$

## Problem 11 (8%).

Let $A$ and $B$ be $n \times n$ matrices with $\det A \neq 0$ and $\det B \neq 0$. Let $O$ be the $n \times n$ matrix with 0 in all entries. Let $I_n$ be the $n \times n$ identity matrix.

Answer the following four true/false problems:

a. It holds that $(AB)^T - B^T A^T = O$.

☒ True ☐ False

b. It holds that $(AB)^{-1} A = B^{-1}$.

☒ True ☐ False

c. It holds that $\det(AB) = 0$.

☐ True ☒ False

d. It holds that $B^{-1} A^{-1} A B = I_n$.

☒ True ☐ False

**Problem 12 (8%).**

Let

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Answer the following three true/false problems:

a. It holds that $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

☒ True ☐ False

b. It holds that $T_B : \mathbb{R}^2 \to \mathbb{R}^2$.

☒ True ☐ False

c. It holds that $T_{AB} : \mathbb{R}^2 \to \mathbb{R}^3$.

☒ True ☐ False

# Delvis facitliste Eksamen 7. januar 2014

**opg. 1**

1. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 5 & 1 \end{bmatrix}$ eller $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 4 & 5 & 1 \end{array}\right]$

3. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

**opg. 2**

1. $\begin{bmatrix} 4 & 1 & 4 \\ 1 & 0 & 1 \\ 5 & 2 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 7 \\ 3 & 3 \end{bmatrix}$

3. "giver ikke mening"

4. "giver ikke mening"

**opg. 3**

1. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

2. $-1$

**opg. 4**

1. Eksempelvis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

2. Eksempelvis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Delvis facitliste eksamen 7. januar 2014

Opg. 5

1. $0, 2$

2. for $\lambda = 0$ eksempelvis $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

for $\lambda = 2$ eksempelvis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Opg. 6

1. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

2. $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Opg. 7

1.

2. $\bar{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\bar{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Opg. 8

1. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$