Exam in Linear Algebra

First Year at The TEK-NAT Faculty and Health Faculty

January 8th, 2013, 9.00-13.00

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts. Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets.

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

COURSE: ☐AAU-Cph (Iver Ottosen)

☐AAU-Esbjerg (Olav Geil, Torben Tvedebrink, Leif K. Jørgensen)

Part I ("regular exercises")

Exercise 1 (10%).

Let

$$A = \begin{bmatrix} 2 & 4 & 4 \\ -3 & -6 & -6 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

1. Reduce A to its reduced row echelon form.
2. Solve the equation $A\mathbf{x} = \mathbf{b}$ or argue that it does not have any solution.
3. Solve the equation $A\mathbf{x} = \mathbf{c}$ or argue that it does not have any solution.

Exercise 2 (6%).

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \text{ and } \mathbf{d} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Determine for each of the following 4 situations if the expression makes sense. Evaluate the expressions that make sense. For the expressions that do not make sense, write "the expression is not defined".

1. $(A\mathbf{c}) + \mathbf{d}$ 2. AB 3. BA 4. $\mathbf{c}^T A$.

Exercise 3 (5%).

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

1. Determine A^{-1}.

Exercise 4 (8%).

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 3x_2 \\ 2x_3 \end{bmatrix}.$$

1. Determine the standard matrix A for T.
2. Determine the standard matrix for the invers linear transformation.

Exercise 5 (10%).

In this exercise we are concerned with a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Consider the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}.$$

The matrix representation of T with respect to $\mathcal{B}$ (also called the $\mathcal{B}$-matrix of T) is given by

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

1. Find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$.
2. Determine the standard matrix A for T.
3. Which geometric operation does T correspond to?

Exercise 6 (10%).

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. Determine for each of the above eigenvalues a basis for the corresponding eigenspace.
3. Is A diagonalizable? (justify your answer).
4. Give an argument that $A^{1057} = A$.

Exercise 7 (12%).

Let $W = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} \right\}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

1. Give an argument that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} \right\}$ is an orthonormal basis for W.
2. Determine $\mathbf{w}$ in W and $\mathbf{z}$ in $W^{\perp}$ such that $\mathbf{u} = \mathbf{w} + \mathbf{z}$.
3. Determine the orthogonal projection matrix P_W.
4. Find a basis for the orthogonal complement $W^{\perp}$.

Exercise 8 (10%).

The subspace W of $\mathbb{R}^5$ has a basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

1. Determine an orthogonal basis for W by using the Gram-Schmidt process.
2. Is the orthogonal basis that you found above an orthonormal basis? (justify your answer).

Part II ("multiple choice" exercises)

Exercise 9 (5%).

The following informations are given. A is a 3×3 symmetric matrix. The vector $\mathbf{u}$ is an eigenvector for A with eigenvalue equal to 2. Furthermore, the vector $\mathbf{v}$ is an eigenvector for A with eigenvalue equal to -1.
Exactly one of the following statements is correct. Tick off the correct answer.

☐ $\mathbf{u} \cdot \mathbf{v} = 0$.
☐ $\mathbf{u} \cdot \mathbf{v} = 2$.
☐ $\mathbf{u} \cdot \mathbf{v} = -2$.

Exercise 10 (6%).

A is an $n \times n$ matrix such that $\det\left(A^3\right) = -27$. For each of the following two questions tick off the correct answer.

$\det(A) =$

☐ -24 ☐ -9 ☐ -3 ☐ 3 ☐ 9 ☐ 24

$\det(A^T A^{-1}) =$

☐ -81 ☐ -9 ☐ -1 ☐ 1 ☐ 9 ☐ 81

Exercise 11 (10%).

Consider

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ -3 \end{bmatrix}.$$

Let $A = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4]$ and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

Tick off correct statements among the ten statements below.

(*Only the statements that you have ticked off will contribute to your mark. Among the statements you have ticked off every incorrect statement will neutralize a correct statement. So if you ticked off 5 statements of which 4 are correct and 1 is wrong, you will receive credits for 4-1=3 correct answers. If you ticked of 4 statements of which 2 are correct and 2 are incorrect, you will receive credits for 2-2=0 correct answer (= no credit). You cannot receive a negative number of points. Hence, if you ticked off 4 statements of which 1 is correct and 3 are incorrect you will receive credits for 0 correct answer (= no credit).*)

- ☐ The dimension of W is 2.
- ☐ The dimension of W is 3.
- ☐ The dimension of W is 4.
- ☐ A is invertible (regular).
- ☐ $\det(A) = 0$.
- ☐ $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a basis for W.
- ☐ $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for W.
- ☐ $\{\mathbf{u}_1, \mathbf{u}_3\}$ is a basis for W.
- ☐ $\{\mathbf{u}_1, \mathbf{u}_4\}$ is a basis for W.
- ☐ $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = W$.

Exercise 12 (8%).

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$ and define the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Answer the following 5 true/false exercises:

a. T is a linear transformation.

☐ True ☐ False

b. T is onto (surjective)

☐ True ☐ False

c. T is one-to-one (injective).

☐ True ☐ False

d. T is invertible.

☐ True ☐ False

e. T corresponds to a rotation.

☐ True ☐ False