Exam in "Linear Algebra"
January 4th, 2011

First Year at The TEK-NAT Faculty and Health Faculty

It is allowed to use books, notes etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam set has two independent parts. Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results. Part II is "multiple choice" exercises. The answers for Part II must be given on these sheets.

Remember to write your full name (including middle names), student number together with the course number on each side of your answers. **Number each page and write the total number of pages on the front page** of the answers.

Good luck!

NAME: ________________________________

STUDENT NUMBER: ________________________________

COURSE NUMBER:

- ☐ Hold 2 (Jacob Broe)
- ☐ Hold 3 (Olav Geil)
- ☐ Hold 4 (Morten Nielsen)
- ☐ Hold 5 (Bo Rosbjerg)
- ☐ Hold 6 (Nikolaj Hess-Nielsen)

Part I ("regular exercises")

Exercise 1 (6%).

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

1. Find A^{-1}.

Exercise 2 (10%).

Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

1. Reduce A to its echelon form.
2. Determine the determinant of A.
3. Determine the determinant of $\left(A^3\right)^T$.

Exercise 3 (10%).

Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

1. Find a basis for the column space associated with A.
2. Find a basis for the null space associated with A.
3. Determine rank A and nullity A.

Exercise 4 (10%).

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for the subspace $W \subseteq \mathbb{R}^4$.

1. Using the Gram-Schmidt procedure, find an orthogonal basis for W.
2. Determine hereafter an orthonormal basis for W.

Exercise 5 (8%).

Let

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. Find a basis for each of the associated eigenspaces.
3. Determine whether A can be diagonalised (Justify your answer).

Exercise 6 (8%).

Let

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}.$$

1. Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\}$ is a basis for W.
2. Justify that $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is in W.
3. Let $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Determine $\mathbf{u}$.

Exercise 7 (8%).

We have that

$$A = \begin{bmatrix} 6 & 6 \\ -2 & -1 \end{bmatrix}$$

can be written as $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix} \quad \text{og} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

1. Find the solution for the system of differential equations

$$\begin{aligned} y_1' &= \ \ 6y_1 + 6y_2 \\ y_2' &= -2y_1 - y_2 \end{aligned}$$

   that satisfies the condition

$$\begin{cases} y_1(0) &= -7 \\ y_2(0) &= 4. \end{cases}$$

Exercise 8 (10%).

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

1. Determine the standard matrix associated with T.
2. Find $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.
3. Is T surjective (onto)?
4. Is T injective (one-to-one)?

Part II ("multiple choice" exercises)

Exercise 9 (4%).

There are given three vectors $\mathbf{u}, \mathbf{v}, \mathbf{z} \in \mathbf{R}^7$, which are linearly *independent*. Let

$$H = \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{z}\}.$$

Mark the true statement below:

- ☐ The dimension of H is 7.
- ☐ The dimension of H is 3.
- ☐ H can be described as a line in $\mathbf{R}^7$.

Exercise 10 (10%).

Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & 1 & 4 & 1 \\ 0 & -15 & 7 & -20 & -5 \\ 0 & 0 & -34 & 50 & -25 \\ 0 & 0 & 0 & -60 & -140 \\ 0 & 0 & 0 & 0 & 910 \end{bmatrix}.$$

Mark *all* the true statements below (note: every wrong answer *eliminates* one correct answer).

- ☐ A is invertible.
- ☐ The linear transformation induced by A is injective (one-to-one).
- ☐ A is on echelon form.
- ☐ nullity $A = 1$.
- ☐ rank $A = 5$.
- ☐ nullity $A +$ rank $A = 6$.
- ☐ The value -15 is an eigenvalue of A.
- ☐ A is on reduced echelon form.
- ☐ There exists a $\mathbf{b} \in \mathbf{R}^5$, such that the system of equations $A\mathbf{x} = \mathbf{b}$ is inconsistent.
- ☐ A is a 4×4-matrix.
- ☐ A can be diagonalised.

Exercise 11 (6%).

There is given a linear mapping $S\colon \mathbf{R}^n \to \mathbf{R}^3$. Answer the following two questions.

Determine the maximal value of n, for which it is true that S it *not* surjective (onto):

□ 0	□ 2	□ 4	□ 6	□ 8	□ 10
□ 1	□ 3	□ 5	□ 7	□ 9	□ 11

Determine the maximal value of n, for which it is true that S *can* be injective (one-to-one):

□ 0	□ 2	□ 4	□ 6	□ 8	□ 10
□ 1	□ 3	□ 5	□ 7	□ 9	□ 11

Exercise 12 (10%).

Answer the following 5 true/false questions:

a. Any symmetric 4×4-matrix can be diagonalised.

☐ True ☐ False

b. There exists a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^5$ which is surjective (onto).

☐ True ☐ False

c. There exists a linear operator $T : \mathbf{R}^3 \to \mathbf{R}^3$, with a orthogonal matrix as standard matrix, such that $T(\mathbf{e}_1) = 4\mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the standard basis for $\mathbf{R}^3$ (i.e. the columns of I_3).

☐ True ☐ False

d. Let W be a subspace of $\mathbf{R}^5$ with dimension 4. Then any orthonormal set of 4 vectors in W constitute a basis for W.

☐ True ☐ False

e. A square matrix A is invertible, if and only if 0 is not an eigenvalue of A.

☐ True ☐ False