Example of a possible exam for Discrete Mathematics

First Year at The TEK-NAT Faculty

???th ?. ????, 20??. Kl. 9-13.

This exam consists of 13 numbered pages with 17 exercises.

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination

This exam has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results
- Part II contains "multiple choice" exercises. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers**

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I: ("Regular exercises")

Exercise 1 (10 %).

Prove by induction that $\sum_{i=1}^{n}(2i) = n(n+1)$, for every positive integer n.

Exercise 2 (10 %).

Consider the set S, $S \subseteq \mathbb{N} \times \mathbb{N}$, recursively defined by

- $(0, 0) \in S$,
- If $(a, b) \in S$ then $(a+1, b+3)$, $(a+2, b+2)$ and $(a+3, b+1)$ are in S.

1. Show that $(4, 8) \in S$.
2. Show by structural induction that 4 divides $a + b$ for all $(a, b) \in S$.

Part II: ("Multiple choice" exercises)

Exercise 3 (4 %).

The set S is defined recursively by

$$1 \in S$$
$$\text{If } x \in S \text{ then } x + 5 \in S \text{ and } x - 5 \in S$$

Which of the following elements is in S? (mark only one answer)

☐ -6 ☐ 0 ☐ 11 ☐ 15

Exercise 4 (6 %).

Let $A = \{a, b, c, d, e\}$ and let $R = \{(a, b), (b, c), (c, d), (b, e)\}$ be a relation on A. Which of the following elements is in the transitive closure R^* of R. (mark only one answer)

☐ (a, d) ☐ (a, a) ☐ (e, a) ☐ (c, a)

The number of elements in the transitive closure of R^* is equal to (mark only one answer)

☐ 4 ☐ 5 ☐ 7 ☐ 8 ☐ 9 ☐ 11

Exercise 5 (6%).

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)(7x^3 + 2x^2)$. One has that $f(x)$ is $\mathcal{O}(x^3)$

a. $(C, k) = (9, 1)$ can be used as witnesses to show that $f(x)$ is $\mathcal{O}(x^3)$

☐ YES ☐ NO

b. $(C, k) = (9, 2)$ can be used as witnesses to show that $f(x)$ is $\mathcal{O}(x^3)$

☐ YES ☐ NO

c. $(C, k) = (9, 0)$ can be used as witnesses to show that $f(x)$ is $\mathcal{O}(x^3)$

☐ YES ☐ NO

d. $(C, k) = (1, 1)$ can be used as witnesses to show that $f(x)$ is $\mathcal{O}(x^3)$

☐ YES ☐ NO

e. $(C, k) = (10, 1)$ can be used as witnesses to show that $f(x)$ is $\mathcal{O}.(x^3)$

☐ YES ☐ NO

f. $(C, k) = (0, 0)$ can be used as witnesses to show that $f(x)$ is $\mathcal{O}(x^3)$

☐ YES ☐ NO

Exercise 6 (6 %).

In this exercise, x is an integer. That is, the domain is $\mathbb{Z}$. Consider the following propositional functions for $x \in \mathbb{Z}$:

$$P(x): \quad ''x \text{ is smaller than } 5'' \tag{1}$$
$$Q(x): \quad ''2 \text{ divides } x'' \tag{2}$$
$$R(x): \quad ''x \text{ is bigger than } 7'' \tag{3}$$

Determine the truth value of the following 6 statements:

a. $P(2)$

☐ True ☐ False

b. $Q(5)$

☐ True ☐ False

c. $P(2) \wedge R(10)$

☐ True ☐ False

d. $\big(P(2) \wedge Q(5)\big) \vee R(12)$

☐ True ☐ False

e. $\forall x(P(x) \vee R(x))$

☐ True ☐ False

f. $\exists x((\neg(P(x) \vee R(x))) \wedge Q(x))$

☐ True ☐ False

Exercise 7 (4 %).

Consider the following algorithm:

Algorithm sum(n: positive integer)
$s := 0$
for $i := 1$ **to** n
  for $j := 1$ **to** n
    s:=s+1
return s

The worst-time complexity of the previous algorithm sum is (mark only one answer)

☐ $\mathcal{O}(n)$ ☐ $\mathcal{O}(n \log n)$ ☐ $\mathcal{O}(n^2)$ ☐ $\mathcal{O}(n^{3/2})$

Exercise 8 (6 %).

Consider the sets $A = \{1, 2\}$ and $B = \{2, 3, 4\}$.

a. Which of the following elements is in $A \times B$? (mark only one answer)

☐ $(1, 1)$ ☐ $(2, 2)$ ☐ $(3, 3)$ ☐ $(4, 1)$

b. What is the cardinality of $A \times B$? (mark only one answer)

☐ 6   ☐ 12   ☐ 16   ☐ 32   ☐ 64

c. What is the cardinality of $\mathcal{P}(A \times B)$? (mark only one answer)

☐ 12   ☐ 16   ☐ 32   ☐ 64   ☐ 128

d. Is $\{(1, 3), (2, 2)\}$ in $\mathcal{P}(A \times B)$? (mark only one answer)

☐ YES   ☐ NO

Exercise 9 (6 %).

Answer the following true/false exercises:

a. One has that $\sum_{i=1}^{7} 1 = 7$?

☐ YES ☐ NO

b. One has that $\sum_{k=0}^{7} k = 8$?

☐ YES ☐ NO

c. One has that $\sum_{s=1}^{n} 1 = n + 1$?

☐ YES ☐ NO

d. One has that $\sum_{i=2}^{4} (i^2 + 1) = 30$?

☐ YES ☐ NO

e. One has that $\sum_{i=1}^{n} i = n(n + 1)/2$?

☐ YES ☐ NO

f. One has that the set $\{\frac{a}{b} \mid a, b$ are positive integers, $a > b\}$ is countable?

☐ YES ☐ NO

Exercise 10 (6 %).

Let $(x+y)^5 = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$, where a, b, c, d, e, f are integers.

a. One has that $b = e$

☐ YES ☐ NO

b. One has that c is equal to (mark only one answer)

☐ 5 ☐ 10 ☐ 20 ☐ -20 ☐ 60

Let $(2x-y)^4 = gx^4 + hx^3y + ix^2y^2 + jxy^3 + ky^4$, where g, h, i, j, k are integers.

c. One has that h is equal to (mark only one answer)

☐ -16 ☐ 16 ☐ 32 ☐ -32

d. One has that j is equal to (mark only one answer)

☐ -8 ☐ 8 ☐ 12 ☐ -12

Exercise 11 (4 %).

a. How many vertices are there in a tree with 12 edges? (mark only one answer)

☐ 11 ☐ 12 ☐ 13 ☐ 24

b. How many edges are there in a tree with 12 vertices? (mark only one answer)

☐ 11 ☐ 12 ☐ 13 ☐ 24

c. How many edges are there in the complete graph K_7 with 7 vertices? (mark only one answer)

☐ 7 ☐ 8 ☐ 20 ☐ 21 ☐ 42

Exercise 12 (9 %).

Consider the graph $G = (V, E)$ with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

a. How many vertices has the graph G? (mark only one answer)

☐ 6 ☐ 12 ☐ 24 ☐ 48

b. How many edges has the graph G? (mark only one answer)

☐ 6 ☐ 8 ☐ 12 ☐ 24

c. Has the graph G a Hamilton circuit?

☐ YES ☐ NO

d. Has the graph G an Euler circuit?

☐ YES ☐ NO

Exercise 13 (8 %).

a. $(603 \cdot 6004 + 60005) \mod 6$ is equal to (mark only one answer)

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

b. $(603 \cdot 6004 + 60005) \mod 10$ is equal to (mark only one answer)

☐ 0 ☐ 2 ☐ 4 ☐ 6 ☐ 8
☐ 1 ☐ 3 ☐ 5 ☐ 7 ☐ 9

Exercise 14 (2 %).

What is the greatest common divisor of 91 and 161?

☐ 1 ☐ 3 ☐ 7 ☐ 13

Exercise 15 (6 %).

Consider the following system of congruences

$$\begin{cases} x \equiv 1 \pmod{3} \\ x \equiv 1 \pmod{4} \\ x \equiv 0 \pmod{5} \end{cases}$$

where we assume that x is non-negative.

a. Is $x = 31$ a solution?

☐ YES ☐ NO

b. Is $x = 25$ a solution?

☐ YES ☐ NO

c. Is there precisely one solution that is smaller than 60?

☐ YES ☐ NO

d. Is there precisely two solutions that are smaller than 120?

☐ YES ☐ NO

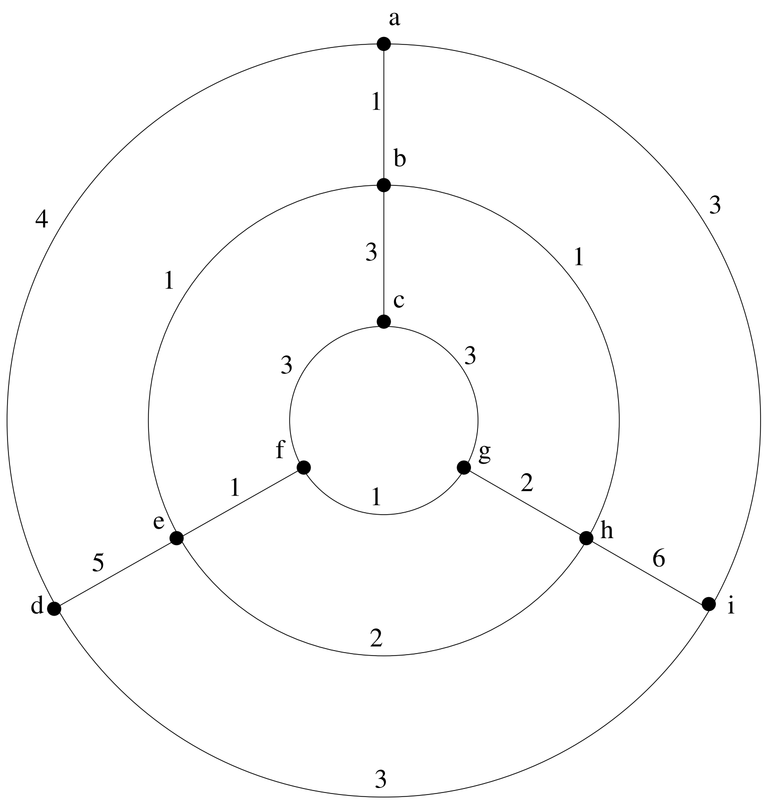

Figure 1:

Exercise 16 (3 %).

What is the weight of the minimum spanning tree of the graph in figure 1?

☐ 12 ☐ 13 ☐ 14 ☐ 15 ☐ 16 ☐ 17

Exercise 17 (4 %).

Suppose that Dijkstra's algorithm is used to determine the length of a shortest path from a to g in the graph in figure 1. Which of the following vertices is added *first* to the set S.

☐ c ☐ d ☐ e ☐ f