Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

August 21, 2015

This exam consists of 10 numbered pages with 17 exercises.

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains "multiple choice" exercises. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

NAME: ____

STUDENT NUMBER: ____

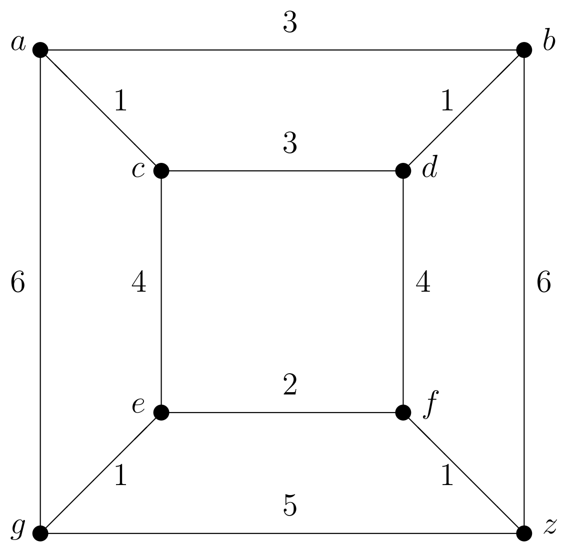

Figure 1:

Part I: (”Regular exercises”)

Exercise 1 (10%)

Prove by induction that

$$\sum_{i=0}^{n}(2 \cdot 3^i) = 3^{n+1} - 1$$

for every integer $n \geq 0$.

Exercise 2 (12%)

Use Dijkstra’s algorithm to determine the length of a shortest path from a to z in the graph shown in Figure 1.
Write the vertices of the graph in an order determined by increasing distance from a.

Part II: (”Multiple choice” exercises)

There is only one correct answer to each question.

Exercise 3 (3%)

What is the weight of a minimum spanning tree of the graph shown in Figure 1?

□ 7 □ 9 □ 11 □ 13 □ 15 □ 17

Exercise 4 (6%)

Let $A = \{\emptyset, \{\emptyset\}, 1, 5\}$ and $B = \{1, 2, 3, 4\}$ be sets.

1. What is the cardinality of $A \cap B$?

   □ 0 □ 1 □ 2 □ 3 □ 4 □ 5 □ 6

2. What is the cardinality of $A \cup B$?

   □ 3 □ 4 □ 5 □ 6 □ 7 □ 8 □ 9

3. What is the cardinality of $A \times B$?

   □ 12 □ 15 □ 16 □ 20 □ 25 □ 30

4. Which of the following is an element of $A \times B$?

   □ $\{\emptyset, 3\}$ □ $\{5, 2\}$ □ $(5, 2)$ □ $(2, 5)$

Exercise 5 (5%)

What is the inverse of 7 modulo 31?

☐ 1 ☐ 9 ☐ 11 ☐ 22 ☐ 23 ☐ 24

Exercise 6 (6%)

Consider the following proposition about Fibonacci numbers $f_0, f_1, f_2, \ldots$:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{1}$$

This proposition can be proved by strong induction. In the basis step it is proved that equation (1) is true for $n = 0$ and for $n = 1$.

Which one of the following statements is a correct beginning of the inductive step (including inductive hypothesis):
(After this we can prove that equation (1) is true for $n = k + 1$.)

☐ Let $k \geq 1$ and suppose that equation (1) is true for $n = k$.
☐ Let $k \geq 2$ and suppose that equation (1) is true for $n = k$.
☐ Let $k \geq 1$ and suppose that equation (1) is true for all n where $0 \leq n \leq k$.
☐ Let $k \geq 2$ and suppose that equation (1) is true for all n where $0 \leq n \leq k$.
☐ Let $k \geq 1$ and suppose that equation (1) is true for all n where $2 \leq n \leq k$.
☐ Let $k \geq 2$ and suppose that equation (1) is true for all n where $2 \leq n \leq k$.

Exercise 7 (4%)

Which one of the following propositions is equivalent to $\neg\forall x\exists y(P(x,y) \vee Q(x,y))$?

☐ $\forall x\exists y(\neg P(x,y) \vee \neg Q(x,y))$
☐ $\exists y\forall x(P(x,y) \wedge Q(x,y))$
☐ $\exists x\forall y(\neg P(x,y) \wedge \neg Q(x,y))$
☐ $\exists x\forall y(\neg P(x,y) \vee \neg Q(x,y))$

Exercise 8 (8%)

Consider the following algorithm:

procedure *matrix-expression*($A = [a_{ij}]$: $n \times n$ matrix)
$s := 0$
for $i := 1$ **to** n
  for $j := 1$ **to** n
    $t := 1$
    for $k := 1$ **to** 3
      $t := a_{ij} \cdot t + 1$
    $s := s + t$
return s

Answer the following two questions about this algorithm:

1. The number mutiplications used by *matrix-expression* is

   ☐ $\Theta(n)$ ☐ $O(n^2)$ ☐ $\Theta(n^2 \log n)$ ☐ $\Omega(n^3)$

2. Which one of the following expressions is computed by *matrix-expression*?

   ☐ $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (a_{ij})^k$

   ☐ $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=0}^{3} (a_{ij})^k$

   ☐ $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{3} (a_{ij})^k$

   ☐ $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=0}^{2} (a_{ij})^k$

Exercise 9 (3%)

Which rule of inference is used in the following argument:

"If we have exam in discrete mathematics then it is Friday. If it is Friday then it is soon weekend. Therefore, if we have exam in discrete mathematics then it is soon weekend."

☐ Conjunction
☐ Modus tollens
☐ Modus ponens
☐ Hypothetical syllogism

Exercise 10 (6%)

A set S of integers is defined recursively by

- $5 \in S$ and $7 \in S$
- if $a \in S$ and $b \in S$ then $a + b$ is also in S.

Answer the following questions about S.

1. What is the largest integer *not* contained in S ?

   ☐ 18 ☐ 19 ☐ 23 ☐ 24 ☐ 29

2. S consists of all numbers of the form $5r + 7t$ where r and t are integers satisfying ...
   What the correct continuation?

   ☐ $r \geq 0 \wedge t \geq 0$
   ☐ $r \geq 1 \wedge t \geq 1$
   ☐ $r \geq 1 \vee t \geq 1$
   ☐ $r \geq 0 \wedge t \geq 0 \wedge r + t > 0$
   ☐ $(r + 1)(t + 1) \geq 2$

Exercise 11 (4%)

Consider the following set of integers

$$S = \{x \mid 0 \leq x < 100 \ \wedge \ x \equiv 3 \pmod 4 \ \wedge \ x \equiv 2 \pmod 5\}.$$

How many integers are there in S?

☐ 0 ☐ 1 ☐ 2 ☐ 5 ☐ 10 ☐ 100

Exercise 12 (7%)

Let $A = \{1, 2, 3, 4, 5\}$ be a set. Consider the following two relations on A

$$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 4), (5, 1), (5, 5)\}$$

$$S = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}.$$

1. Answer the following true/false exercises:

   a. R is transitive

   ☐ True ☐ False

   b. R is reflexive

   ☐ True ☐ False

   c. R is symmetric

   ☐ True ☐ False

   d. S is antisymmetric

   ☐ True ☐ False

   e. $(5, 2)$ is in the composed relation $R \circ S$

   ☐ True ☐ False

   f. $(5, 2)$ is in the composed relation $S \circ R$

   ☐ True ☐ False

2. Let S^* denote the transitive closure of S. How many pairs (a, b) are there in S^* ?

   ☐ 5 ☐ 10 ☐ 15 ☐ 20 ☐ 25

Exercise 13 (5%)

a. How many rows appear in a truth table of the compound proposition

$$(p \wedge q) \vee (r \wedge \neg p)$$

☐ 1 ☐ 2 ☐ 4 ☐ 6 ☐ 8 ☐ 16

b. $(p \wedge q) \vee (r \wedge \neg p)$ is a tautology.

☐ True ☐ False

c. The propositions $(p \wedge q) \vee (q \wedge \neg p)$ and $(p \vee \neg p) \wedge q$ are equivalent.

☐ True ☐ False

Exercise 14 (4%)

Consider the graph G in Figure 2.

a. What is degree of the vertex v

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

b. What is the number of edges in a spanning tree of G

☐ 6 ☐ 7 ☐ 8 ☐ 9 ☐ 14 ☐ 17

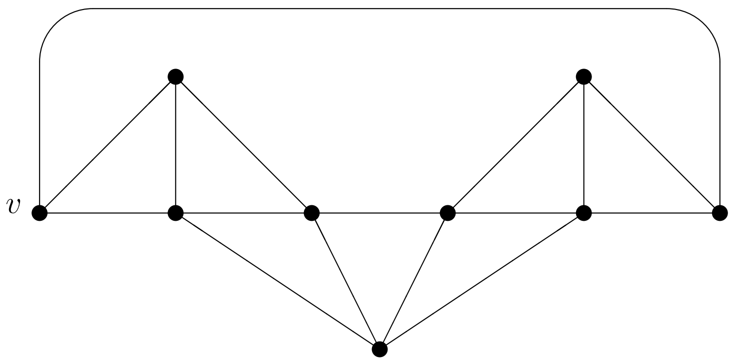

Figure 2: The graph G considered in Exercises 14 and 15.

Exercise 15 (6%)

Consider the graph G in Figure 2.
Answer the following true/false questions.

a. G is a simple graph.

☐ True ☐ False

b. G is connected.

☐ True ☐ False

c. G has an Euler circuit.

☐ True ☐ False

d. G has a Hamilton circuit.

☐ True ☐ False

e. G has a Hamilton path.

☐ True ☐ False

Exercise 16 (6%)

Let $f(x) = 5x^2 + 4x + 3$. One has that $f(x)$ is $O(x^2)$.

a. $(C, k) = (12, 0)$ can be used as witnesses to show that $f(x)$ is $O(x^2)$.

☐ True ☐ False

b. $(C, k) = (10, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^2)$.

☐ True ☐ False

c. $(C, k) = (12, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^2)$.

☐ True ☐ False

d. $(C, k) = (29, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^2)$.

☐ True ☐ False

e. $(C, k) = (7, 2)$ can be used as witnesses to show that $f(x)$ is $O(x^2)$.

☐ True ☐ False

f. $(C, k) = (9, 2)$ can be used as witnesses to show that $f(x)$ is $O(x^2)$.

☐ True ☐ False

Exercise 17 (5%)

Suppose we have to sort 1000 numbers in an increasing order. We can use either the algorithm *Bubble Sort* or the algorithm *Merge Sort*. Which of the following claims is correct?

☐ *Merge Sort* uses more comparisons than *Bubble Sort*.
☐ The two algorithms use approximately the same number of comparisons.
☐ The number of comparisons used by *Bubble Sort* is approximately 3 times the number of comparisons used by *Merge Sort*.
☐ The number of comparisons used by *Bubble Sort* is approximately 50 times the number of comparisons used by *Merge Sort*.