Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

June 15th, 2015, 9.00-13.00

This exam consists of 12 numbered pages with 17 exercises.

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam has two independent parts.

- Part I contains "regular exercises". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains "multiple choice" exercises. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I: ("Regular exercises")

Exercise 1 (8%)

A sequence $a_0, a_1, a_2, a_3, \ldots$ of integers is defined recursively by

$a_0 = 1,$
$a_{n+1} = 2a_n - n,$ for all $n \geq 0.$

Prove by induction that $a_n = n + 1$ for all $n \geq 0$.

Exercise 2 (9%)

Consider the following algorithm.

procedure *sum*(n: positive integer)
$i := 1$
$x := 1$
$s := 1$
while $i < n$
  $i := i + 1$
  $x := x + 2$
  $s := s + x$
return s

1. Prove that the following assertion is a loop invariant for the while-loop:

$$i \in \mathbb{N} \ \wedge \ i \leq n \ \wedge \ x = 2i - 1 \ \wedge \ s = i^2. \qquad (1)$$

2. What is the value of s in terms of n when the algorithm terminates? Justify your answer.

Part II: ("Multiple choice" exercises)

There is only one correct answer to each question.

Exercise 3 (3%)

Using the extended Euclidean algorithm we find that

$$\gcd(258, 369) = -10 \cdot 258 + 7 \cdot 369 = 3.$$

Which one of the following statements is true?

☐ -10 is an inverse of 258 modulo 369.

☐ 359 is an inverse of 258 modulo 369.

☐ 7 is an inverse of 258 modulo 369.

☐ 258 has no inverse modulo 369.

Exercise 4 (4%)

Which one of the following sets is *not* countable?

☐ The set of prime numbers
☐ $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$
☐ $\mathbb{Z} \times \mathbb{Z}$
☐ $\{x \in \mathbb{Q} \mid -3 \leq x \leq \sqrt{2}\}$

Exercise 5 (4%)

Which one of the following propositions is equivalent to $\forall x \exists y P(x, y)$?

☐ $\forall y \exists x P(x, y)$ ☐ $\exists x \forall y P(x, y)$ ☐ $\forall y \exists x P(y, x)$ ☐ $\exists y \forall x P(x, y)$

Exercise 6 (8%)

Consider the Merge Sort algorithm on page 360 in [Rosen, Discrete Mathematics and its Applications, Seventh Edition, Global Edition] using procedure *merge* on page 361.
Let $P(x, y)$ denote the statement: "if we use Merge Sort and procedure merge to sort the list

$$5, 2, 7, 3, 6, 1, 9, 4$$

then at some step we will directly compare x and y."
What is the truth value of each of the following propositions:

a. $P(3, 6)$

☐ True ☐ False

b. $P(3, 4)$

☐ True ☐ False

c. $P(2, 7)$

☐ True ☐ False

d. $P(1, 4)$

☐ True ☐ False

Exercise 7 (5%)

a. The propositions $r \to s$ and $\neg r \vee s$ are equivalent.

☐ True ☐ False

b. How many rows appear in a truth table of the compound proposition

$$p \vee \neg q \leftrightarrow \neg p \vee q$$

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 6 ☐ 8

c. In how many rows in this truth table is the truth value of $p \vee \neg q \leftrightarrow \neg p \vee q$ "true (T)" ?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

d. $p \vee \neg q \leftrightarrow \neg p \vee q$ is a tautology.

☐ True ☐ False

Exercise 8 (3%)

Which rule of inference is used in the following argument:

"If it is Valdemar's day then there are flags on the buses. It is Valdemar's day. Therefore, there are flags on the buses."

☐ Conjunction
☐ Modus tollens
☐ Modus ponens
☐ Universal generalization

Exercise 9 (4%)

Consider the following set of integers

$$S = \{x \mid 0 \leq x < 280 \ \wedge \ x \equiv 3 \pmod 7 \ \wedge \ x \equiv 4 \pmod 8\}.$$

How many integers are there in S?

☐ 0 ☐ 1 ☐ 2 ☐ 5 ☐ 10 ☐ 280

Exercise 10 (5%)

Let $(x-y)^5 = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$, where a, b, c, d, e, f are integers.

a. One has that $b = e$

☐ YES ☐ NO

b. One has that d is equal to

☐ 5 ☐ 10 ☐ 20 ☐ -5 ☐ -10 ☐ -20

Let $(3x+2y)^3 = gx^3 + hx^2y + ixy^2 + jy^3$, where g, h, i, j are integers.

c. One has that h is equal to

☐ 6 ☐ 18 ☐ 27 ☐ 36 ☐ 54 ☐ 81

Exercise 11 (7%)

Let $A = \{1, 2, 3, 4, 5\}$ be a set. Consider the following two relations on A

$$S = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,1), (4,3), (4,4)\}$$

$$R = \{(2,1), (2,3), (4,1), (4,3), (4,4), (4,5), (5,1)\}$$

Answer the following true/false exercises:

a. R is transitive

☐ True ☐ False

b. R is reflexive

☐ True ☐ False

c. S is an equivalence relation

☐ True ☐ False

d. $(1, 3)$ is in the transitive closure of S

☐ True ☐ False

e. $(2, 5)$ is in the transitive closure of S

☐ True ☐ False

f. $(3, 5)$ is in the composed relation $R \circ S$

☐ True ☐ False

g. $(3, 5)$ is in the composed relation $S \circ R$

☐ True ☐ False

Exercise 12 (7%)

$(111 \cdot 11113 + 1111115) \mod 11$ is equal to

☐ 0	☐ 1	☐ 2	☐ 3	☐ 4	☐ 5
☐ 6	☐ 7	☐ 8	☐ 9	☐ 10	

Exercise 13 (9%)

Let $f(x) = (x \log x + 5x)(x^2 + 3x - 4)$, for $x > 0$.
Answer the following 6 true/false exercises.

a. $f(x)$ is $O(x^3)$

☐ True ☐ False

b. $f(x)$ is $O(x^4)$

☐ True ☐ False

c. $f(x)$ is $O(x^3 \log x)$

☐ True ☐ False

d. $f(x)$ is $\Theta(x^3 \log x)$

☐ True ☐ False

e. $f(x)$ is $\Omega(x^3)$

☐ True ☐ False

f. $f(x)$ is $O(x^2 \log x)$

☐ True ☐ False

Exercise 14 (6%)

Let $f(x) = 3x^3 + 2x + 4$. One has that $f(x)$ is $O(x^3)$.

a. $(C, k) = (10, 0)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☐ False

b. $(C, k) = (6, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☐ False

c. $(C, k) = (9, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☐ False

d. $(C, k) = (12, 1)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

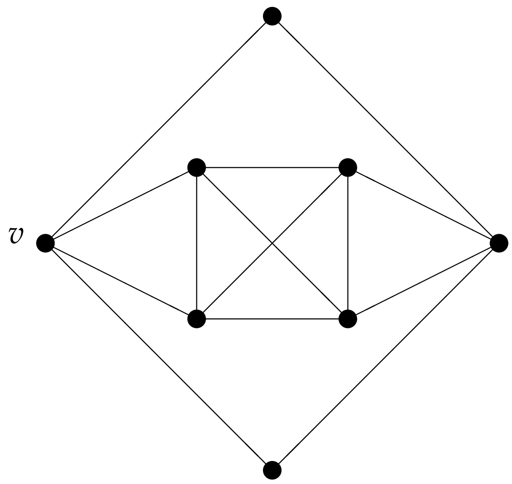
☐ True ☐ False

e. $(C, k) = (3, 2)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☐ False

f. $(C, k) = (5, 2)$ can be used as witnesses to show that $f(x)$ is $O(x^3)$.

☐ True ☐ False



Figur 1: The graph G considered in Exercises 16 and 17.

Exercise 15 (6%)

Let $A = \{\emptyset, 1, 2, 3, 4\}$ and $B = \{\{\emptyset\}, 2, 4, 6\}$ be sets.

1. What is the cardinality of $A \cap B$?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

2. What is the cardinality of $A \cup B$?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ☐ 8

3. What is the cardinality of $A \times B$?

   ☐ 12 ☐ 15 ☐ 16 ☐ 20 ☐ 25 ☐ 30

4. Which one of the follow is an element of $A \times B$?

   ☐ $\{\emptyset, \emptyset\}$ ☐ $(\emptyset, \emptyset)$ ☐ $(\emptyset, \{\emptyset\})$ ☐ $(\{\emptyset\}, 6)$

Exercise 16 (6%)

Consider the graph G in Figure 1.
Answer the following true/false questions.

a. G is a simple graph.

☐ True ☐ False

b. G is connected.

☐ True ☐ False

c. G has an Euler circuit.

☐ True ☐ False

d. G has a Hamilton circuit.

☐ True ☐ False

d. G has a Hamilton path.

☐ True ☐ False

Exercise 17 (6%)

Consider again the graph G in Figure 1.

a. What is degree of the vertex v

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

b. What is the largest number of vertices in a complete subgraph of G

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

c. What is the length of a shortest *simple* circuit of G

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

d. What is the number of edges in a spanning tree of G

☐ 0 ☐ 1 ☐ 6 ☐ 7 ☐ 8 ☐ 14