Exam in Discrete Mathematics

First Year at The TEK-NAT Faculty

June 11th, 2014, 9.00–13.00

It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual exercises influence the total examination.

This exam has two independent parts.

- Part I contains ”regular exercises”. Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II contains ”multiple choice”exercises. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each page of your answers. **Number each page and write the total number of pages on the front page of the answers.**

NAME: ____

STUDENT NUMBER: ____

Part I ("regular exercises")

Exercise 1 (6%).

Find the expansion of $(2x - y)^4$ using The Binomial Theorem.

Exercise 2 (8%).

Find witnesses proving that $f(x) = 2x^3 + x^2 + 5$ is $O(x^3)$.

Exercise 3 (12%).

1. Use the Euclidean algorithm to find the greatest common divisor of 46 and 21.
2. Find integers s and t satisfying that $\gcd(46, 21) = s \cdot 46 + t \cdot 21$.
3. Determine all integers x such that

$$x \equiv 2 \pmod{46} \quad \text{and} \quad x \equiv 1 \pmod{21}.$$

Exercise 4 (9%).

Prove by induction that

$$\sum_{i=1}^{n} (4i + 1) = 2n^2 + 3n,$$

for every positive integer n.

Exercise 5 (6%).

1. Construct a truth table for the compound proposition $(p \wedge \neg q) \rightarrow (r \vee q)$.
2. Is the compound proposition in question 1 a tautology?

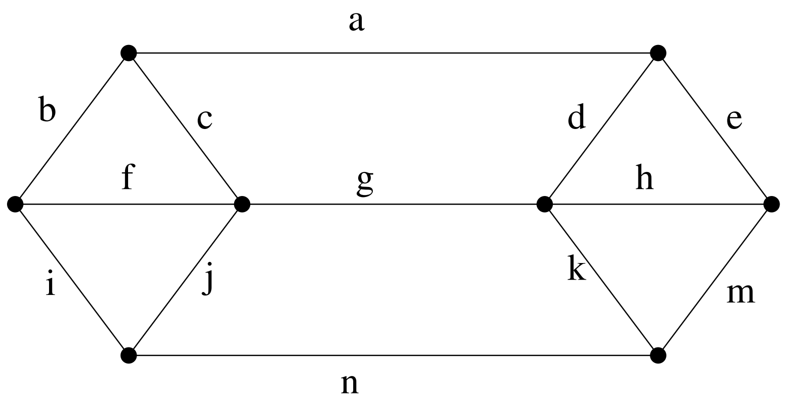

Figure 1: A graph G considered in Exercise 6.

Exercise 6 (10%).

A graph G with 13 edges is shown in Figure 1. The edges of G have weights given by the following table

Edge	a	b	c	d	e	f	g	h	i	j	k	m	n
Weight	1	1	3	3	6	4	5	6	2	4	2	7	2

1. Use Prim's algorithm to find a minimum spanning tree S in G. Write the edges of S in the order in which they are added to S by Prim's algorithm. (If there is more than one possible solution then write only one of them.)
2. Use Kruskal's algorithm to find a minimum spanning tree T in G. Write the edges of T in the order in which they are added to T by Kruskal's algorithm. (If there is more than one possible solution then write only one of them.)

Exercise 7 (9%).

Let $A = \{a, b, c, d\}$ and let $R = \{(a, b), (b, c), (c, d), (d, b)\}$ be a relation on A.

1. Draw the directed graph representing R.
2. Determine the transitive closure R^* of R.
3. Determine a matrix $\mathbf{M}_{R^*}$ representing R^*.

Exercise 8 (10%).

A set S is defined recursively by

Basis step: $0 \in S$

Recursive step: if $a \in S$ then $a + 3 \in S$ and $a + 5 \in S$.

1. Determine the set $S \cap \{a \in \mathbb{Z} \mid 0 < a < 12\}$.
2. Prove that every integer $a \geq 8$ is contained in S.

Part II ("multiple choice" exercises)

Exercise 9 (10%).

Let $f(x) = (x^2 + 5x + 3)(x + 2\log x)$, for $x > 0$. Answer the following 5 true/false exercises

1. $f(x)$ is $O(x^4)$.

   ☐ True ☐ False

2. $f(x)$ is $O(x^3)$.

   ☐ True ☐ False

3. $f(x)$ is $O(x^2)$.

   ☐ True ☐ False

4. $f(x)$ is $O(x^3 \log x)$.

   ☐ True ☐ False

5. $f(x)$ is $O(x^2 \log x)$.

   ☐ True ☐ False

Exercise 10 (6%).

Let $A = \{1, 3, 5\}$ and $B = \{3, 4, 5\}$ be sets.

1. What is the cardinality of the power set $\mathcal{P}(A \cup B)$

   ☐ 4 ☐ 8 ☐ 16 ☐ 32 ☐ 64

2. Which of the following are elements of $A \times B$?

   ☐ $\{1, 3\}$ ☐ $(1, 3)$ ☐ $(4, 5)$ ☐ $(5, 5)$

Exercise 11 (8%).

Consider the following algorithm:

Procedure sum(n: positive integer)
$s := 0$
for $i := 1$ **to** n
  for $j := 1$ **to** i
    $s := s + j$
return s

1. Suppose that procedure sum is started with input $n = 4$. Then what number is returned by the algorithm?

   ☐ 10 ☐ 20 ☐ 40 ☐ 45

2. The worst-case time complexity of procedure sum is:

   ☐ $O(n)$ ☐ $O(n \log n)$ ☐ $O(n^{3/2})$ ☐ $O(n^2)$

Exercise 12 (6%).

Let

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

be a matrix representing a relation R on a set A. Answer the following 3 true/false exercises

1. R is reflexive.

   ☐ True ☐ False

2. R is symmetric.

   ☐ True ☐ False

3. R is antisymmetric.

   ☐ True ☐ False