Lecture Notes on Polynomials

Arne Jensen
Department of Mathematical Sciences
Aalborg University

Third edition
Revised March, 2014

1 Introduction

These lecture notes give a very short introduction to polynomials with real and complex coefficients.

2 Definitions and Some Properties

Polynomials with complex coefficients are functions of a complex variable z of a particularly simple form. Examples are

$$z^2 + (8+i)z + 4, \quad z^{16} - 64, \quad (7-8i)z^3 - (4+4i)z^2 - \sqrt{17}, \quad 232, \quad \text{and} \quad z-1. \tag{2.1}$$

The formal definition is as follows.

Definition 2.1. A polynomial with complex coefficients is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{2.2}$$

where $a_j \in \mathbf{C}$, $j = 0, 1, \ldots, n$, and z is a complex variable. If $a_n \neq 0$, then n is the *degree* of $p(z)$, which is written as $\deg(p(z)) = n$. In general, the degree of a polynomial $p(z)$ is the largest k such that $a_k \neq 0$. The polynomial with all coefficients equal to zero is called the *zero polynomial*. The degree of the zero polynomial is defined to be zero[1].

Looking at the examples in (2.1), we see that the degree of the first polynomial is 2, the second one has degree 16, etc.

A number of operations can be performed with polynomials. Given a polynomial $p(z)$ and a complex number c, the polynomial $c\,p(z)$ is obtained by multiplying each coefficient in $p(z)$ by c. Given two polynomials $p(z)$ and $q(z)$, their sum is defined by adding the coefficients of corresponding power. Some examples will illustrate these definitions.

[1] This choice requires some care in certain computations, which however will not be needed there. In most cases one prefers to assign the degree $-\infty$ to the zero polynomial, but that also requires some care in computations

Given $p(z) = z^3 - iz + 1 + 7i$ and $c = 1 - i$, we have

$$\begin{aligned} c\,p(z) &= (1-i)(z^3 - iz + 1 + 7i) = (1-i)z^3 + (1-i)(-i)z + (1-i)(1+7i) \\ &= (1-i)z^3 + (-1-i)z + 8 + 6i. \end{aligned}$$

Let $q(z) = -z^3 + 4z^2 - 8z - 8$. Then the polynomial $p(z) + q(z)$ is given by

$$\begin{aligned} p(z) + q(z) &= (z^3 - iz + 1 + 7i) + (-z^3 + 4z^2 - 8z - 8) \\ &= (1 + (-1))z^3 + (0+4)z^2 + (-i + (-1-i))z + (1 + 7i + (-8)) \\ &= 4z^2 + (-1-2i)z - 7 + 7i. \end{aligned}$$

Note that since the coefficient to the term z^2 in $p(z)$ is zero, it is not written explicitly in the usual expression for $p(z)$, but we have included it in the computation above to clarify the principle of addition.

Polynomials can be multiplied. Given $p_1(z) = z^2 - i$ and $q_1(z) = z^3 - z$, the product is obtained by multiplying out and collecting coefficients to the same power of z. We have

$$\begin{aligned} p_1(z)q_1(z) &= (z^2 - i)(z^3 - z) = z^2(z^3 - z) - i(z^3 - z) \\ &= z^5 - z^3 - iz^3 + iz = z^5 + (-1-i)z^3 + iz. \end{aligned}$$

In general, we cannot divide polynomials and obtain a quotient, which is again a polynomial. But division with remainder can be carried out. The method is the same as used for integers. Given the integers $m = 9$ and $n = 4$, division of m by n with remainder means that we can write $m = kn + r$, where k is an integer, and the remainder r is an integer that satisfies $0 \leq r < n$. Thus the result in the example is $9 = 2 \cdot 4 + 1$. The assumption needed to carry out this division with remainder is that $m \geq n$.

Given two polynomials $p_1(z)$ and $p_2(z)$, such that $\deg(p_1) \geq \deg(p_2) > 0$, division with remainder means to write

$$p_1(z) = q(z)p_2(z) + r(z).$$

Here $q(z)$ is a polynomial and $r(z)$ is a polynomial satisfying $0 \leq \deg(r) < \deg(p_2)$.

Here are some examples to illustrate this procedure. First let us take $p_1(z) = z^4 - z^3 + z^2 - z$ and $p_2(z) = z^2 - 1$. Then the result is

$$z^4 - z^3 + z^2 - z = (z^2 - z + 2)(z^2 - 1) + (-2z + 2),$$

such that $q(z) = z^2 - z + 2$ and $r(z) = -2z + 2$.

For the next example we take $p_1(z) = 4z^4 - 64$ and $p_2(z) = z^2 - 4$. In this case

$$4z^4 - 64 = (4z^2 + 16)(z^2 - 4) + 0.$$

Thus in this case the remainder is zero.

There are various ways of doing these divisions with remainder by hand. At least one of the methods will be illustrated during the lectures. It is also possible to use Maple to carry out the computation of the quotient and the remainder. The functions are called `quo` and `rem`, respectively. See the Maple documentation for their use.

3 Roots of Polynomials

We introduce the following definition.

Definition 3.1. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. A complex number $z_0 \in \mathbf{C}$ is called a *root* of $p(z)$, if $p(z_0) = 0$.

Thus a root of the polynomial $p(z)$ is just a different name for a *zero* of $p(z)$ as a function. The reason for using a special name is that roots of a polynomial have many nice properties not shared by zeroes of general functions.

We have the following important result.

Proposition 3.2. *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then $z_0 \in \mathbf{C}$ is a root of $p(z)$, if and only if there exists a polynomial $q(z)$ (of degree $n-1$), such that*

$$p(z) = q(z)(z - z_0). \tag{3.1}$$

Proof. If (3.1) holds, then it is obvious that $p(z_0) = 0$. Conversely, assume that z_0 is a root of $p(z)$. Then we can use the division with remainder described in the previous section to write $p(z) = q(z)(z - z_0) + c$, where c is the remainder, a polynomial of degree 0. Now if we use that $p(z_0) = 0$, it follows that $c = 0$ and (3.1) holds. □

Definition 3.3. Let $p(z)$ be a polynomial of degree $n \geq 1$. Assume that z_0 is a root of $p(z)$. We define the *multiplicity* of the root z_0 to be the integer m that satisfies

$$p(z) = q(z)(z - z_0)^m \quad \text{and} \quad q(z_0) \neq 0. \tag{3.2}$$

The most important result about polynomials is the following result, which is called the Fundamental Theorem of Algebra. This theorem is not easy to prove, so we will state it without proof.

Theorem 3.4 (Fundamental Theorem of Algebra)**.** *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then $p(z)$ always has a root $z_0 \in \mathbf{C}$.*

It is important to note that this theorem states that there always exists a root in any polynomial of degree greater than or equal to one. But the theorem does not give a method or an algorithm to find a root. Actually there is no general algorithm to find the exact roots of a polynomial of degree five or higher.

One can apply the Fundamental Theorem of Algebra repeatedly to obtain the following result.

Corollary 3.5. *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then there exist complex numbers $z_1, z_2, \ldots, z_n$, such that*

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n). \tag{3.3}$$

Proof. We use the Fundamental Theorem of Algebra to write $p(z) = q_1(z)(z - z_1)$ for some complex number z_1. Now $q_1(z)$ is a polynomial of degree $n-1$. If $n-1 > 0$, we can apply the Fundamental Theorem of Algebra once more to write $q_1(z) = q_2(z)(z - z_2)$ for some complex number z_2. Repeating this argument the result follows. □

We can use Definition 3.3 and Corollary 3.5 to obtain the following result, by grouping together repeated roots in (3.3).

Corollary 3.6. *Let $p(z)$ be a polynomial of degree $n \geq 1$. Then there exist complex numbers $\zeta_1, \zeta_2, \cdots, \zeta_k$, $\zeta_j \neq \zeta_{j'}$, $j \neq j'$, and integers $m_1, m_2, \cdots, m_k$, satisfying $1 \leq m_j \leq n$, $j = 1, 2, \ldots, k$ and $m_1 + m_2 + \cdots + m_k = n$, such that*

$$p(z) = a_n(z - \zeta_1)^{m_1}(z - \zeta_2)^{m_2} \cdots (z - \zeta_k)^{m_k}. \tag{3.4}$$

We note that m_j is the multiplicity of the root ζ_j.

We conclude this section with a few examples of factorizations. We consider first $p(z) = z^4 + 2z^2 + 1$. We have $p(z) = (z - i)^2(z + i)^2$. Thus this polynomial has two different complex roots, $+i$ and $-i$, and each of these roots has multiplicity 2.

Next we take $p(z) = 6\,z^3 - 6i\,z^2 + 12\,z - 6\,z^2 + 6i\,z - 12$. In this case one can show that $p(z) = 6(z-1)(z+i)(z-2i)$. Thus the roots are 1, $-i$, and $2i$, and all three roots have multiplicity one.

4 Roots in Polynomials of Degree One and Two

Let us start with the easy case of a polynomial of degree one, $p(z) = a_1 z + a_0$, $a_1 \neq 0$. The root is given by $z_1 = -a_0/a_1$ and has multiplicity one.

Next we look at a special type of polynomial of degree two, $p(z) = z^2 - a$. We have the following result.

Proposition 4.1. *Let $p(z) = z^2 - a$, where $a = \alpha + i\beta$, $\alpha, \beta \in \mathbf{R}$. Let*

$$\operatorname{sgn}(\beta) = \begin{cases} 1, & \text{if } \beta \geq 0, \\ -1, & \text{if } \beta < 0. \end{cases} \tag{4.1}$$

Let $r = |a| = \sqrt{\alpha^2 + \beta^2}$. Then the two roots of $p(z) = z^2 - a$ are given by

$$z_1 = \sqrt{\frac{r+\alpha}{2}} + i \operatorname{sgn}(\beta) \sqrt{\frac{r-\alpha}{2}}, \tag{4.2}$$

$$z_2 = -\sqrt{\frac{r+\alpha}{2}} - i \operatorname{sgn}(\beta) \sqrt{\frac{r-\alpha}{2}}. \tag{4.3}$$

Proof. The proof is very simple. One needs to verify that $(z_1)^2 = a$ and $(z_2)^2 = a$. Let us verify the first equality. We have

$$\begin{aligned} (z_1)^2 &= \left(\sqrt{\frac{r+\alpha}{2}} + i \operatorname{sgn}(\beta) \sqrt{\frac{r-\alpha}{2}} \right)^2 \\ &= \tfrac{1}{2}(r+\alpha) - \tfrac{1}{2}(r-\alpha) + 2i \operatorname{sgn}(\beta) \sqrt{\tfrac{1}{4}(r+\alpha)(r-\alpha)} \\ &= \alpha + i \operatorname{sgn}(\beta) \sqrt{\beta^2} = \alpha + i \operatorname{sgn}(\beta) |\beta| \\ &= \alpha + i\beta = a. \end{aligned}$$

In this computation we have used that $(r+\alpha)(r-\alpha) = r^2 - \alpha^2 = (\alpha^2 + \beta^2) - \alpha^2 = \beta^2$ and $\beta = \operatorname{sgn}(\beta)|\beta|$. □

Based on this result we can now find the roots in a general polynomial of degree two. We have the following result.

Proposition 4.2. *Let $p(z) = az^2 + bz + c$, where $a, b, c \in \mathbf{C}$ with $a \neq 0$. Let $D = b^2 - 4ac$, and let w be one of the solutions to $z^2 - D = 0$. Then the roots of $p(z)$ are given by*

$$\frac{-b \pm w}{2a}. \tag{4.4}$$

D is called the discriminant *of the polynomial.*

Proof. Since $a \neq 0$, we can rewrite the polynomial $p(z)$ as follows.

$$\begin{aligned} p(z) &= az^2 + bz + c = a\big(z^2 + \frac{b}{a}z + \frac{c}{a}\big) \\ &= a\Big(\big(z + \frac{b}{2a}\big)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\Big) \\ &= a\Big(\big(z + \frac{b}{2a}\big)^2 - \frac{D}{4a^2}\Big) \\ &= a\Big(\big(z + \frac{b}{2a}\big)^2 - \frac{w^2}{4a^2}\Big) \\ &= a\big(z + \frac{b}{2a} - \frac{w}{2a}\big)\big(z + \frac{b}{2a} + \frac{w}{2a}\big), \end{aligned}$$

which shows that the two roots are given by (4.4). □

We now give a few examples of the use of these two results. First we solve the equation $z^2 = 2 - 2i$. Thus we have $\alpha = 2$, $\beta = -2$, $r = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$ and $\operatorname{sgn}(\beta) = -1$. Thus

$$z_1 = \sqrt{\tfrac{1}{2}(2\sqrt{2}+2)} + i(-1)\sqrt{\tfrac{1}{2}(2\sqrt{2}+2)} = \sqrt{\sqrt{2}+1} - i\sqrt{\sqrt{2}-1}.$$

The other root is of course $z_2 = -z_1$.

Next we find the roots of the polynomial $2\,z^2 - 10i\,z - 12$. First we compute the discriminant:

$$D = (-10i)^2 - 4 \cdot 2 \cdot (-12) = -4.$$

One of the solutions to $w^2 = -4$ is $w = 2i$. Thus the roots are

$$\frac{-(-10i) \pm 2i}{2 \cdot 2} = \begin{cases} 3i, \\ 2i. \end{cases}$$

Remark 4.3. There is an easy way to test whether one has found the correct roots of a polynomial of degree two. Assume for simplicity that $a = 1$, such that we have the roots z_1 and z_2 of the polynomial $z^2 + bz + c$. These two roots must then satisfy

$$z_1 + z_2 = -b \quad \text{and} \quad z_1 z_2 = c. \tag{4.5}$$

The verification of these results is left to the reader.

5 Roots of $z^m - a$

In this section we review the results from [1] concerning roots of the polynomial $z^m - a$, or equivalently, solutions to the equation $z^m = a$, where $m \geq 1$ is an integer. The method is to write a in polar form

$$a = re^{i\theta}, \quad r = |a|, \quad \theta = \operatorname{Arg} a.$$

The m different solutions are then given by

$$z_k = r^{1/m}\big(\cos(\frac{\theta + 2\pi k}{m}) + i\sin(\frac{\theta + 2\pi k}{m})\big), \quad k = 0, 1, \ldots, m-1. \tag{5.1}$$

Examples and further comments can be found in [1].

6 Factorization of Polynomials

The results in Corollaries 3.5 and 3.6 show that once we know the roots in a polynomial, then we can factor it. Even for polynomials with real coefficients we may get non-real numbers in the factorization, as in

$$4\,z^2 + 16 = 4(z - 2i)(z + 2i).$$

However, if we are satisfied with a factorization in factors that are of degree one or two, then it can be achieved with real coefficients only. Before we state this result, we need the following important result.

Proposition 6.1. *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with* real *coefficients $a_j \in \mathbf{R}$, $j = 0, 1, 2, \ldots, n$. If z_0 is a root of $p(z)$, then the conjugate $\overline{z_0}$ is also a root of $p(z)$.*

Proof. We have $p(z_0) = a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0 = 0$. Taking the complex conjugate we get, using the facts that the conjugate of a sum is the sum of the sum of the conjugates, and the conjugate of a product is the product of the conjugate of each factor,

$$\begin{aligned}
\overline{p(z_0)} &= \overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0} \\
&= \overline{a_n z_0^n} + \overline{a_{n-1} z_0^{n-1}} + \cdots + \overline{a_1 z_0} + \overline{a_0} \\
&= a_n \overline{z_0}^n + a_{n-1} \overline{z_0}^{n-1} + \cdots + a_1 \overline{z_0} + a_0 \\
&= p(\overline{z_0}).
\end{aligned}$$

In the computation above we have used that the coefficients a_j are real, such that $\overline{a_j} = a_j$, $j = 0, 1, \ldots, n$. Thus $p(z_0) = 0$ implies $p(\overline{z_0}) = 0$. □

Proposition 6.2. *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with* real *coefficients $a_j \in \mathbf{R}$, $j = 0, 1, 2, \ldots, n$. Let ξ_j, $j = 1, 2, \ldots, J$ denote the distinct real roots of $p(z)$, and let ζ_k, $k = 1, 2, \ldots, K$ be complex numbers with $\operatorname{Im} \zeta_k \neq 0$, such that $\zeta_k, \overline{\zeta_k}$ are the remaining distinct roots of $p(z)$, $k = 1, \ldots, K$. Write $\zeta_k = \alpha_k + i\beta_k$ with α_k and β_k real. Then we have*

$$p(z) = a_n (z - \xi_1)^{n_1} \cdots (z - \xi_J)^{n_J} ((z - \alpha_1)^2 + \beta_1^2)^{m_1} \cdots ((z - \alpha_K)^2 + \beta_K^2)^{m_K}. \tag{6.1}$$

Here n_j is the multiplicity of the root ξ_j, $j = 1, \ldots, J$ and m_k is the multiplicity of the root ζ_k, $k = 1, \ldots, K$. We have $J + 2K = n$.

Proof. The result is a consequence of Corollary 3.6 and Proposition 6.1. For the real roots this is immediate. For the pairs of complex conjugate roots we use that

$$(z - \zeta_k)(z - \overline{\zeta_k}) = (z - \alpha_k - i\beta_k)(z - \alpha_k + i\beta_k) = (z - \alpha_k)^2 + \beta_k^2.$$

□

Let us give some examples. First we consider $p(z) = z^4 - 1$. Here we can use the result from Section 5 to find the roots. The roots are $z_1 = 1$, $z_2 = -1$, $z_3 = i$, and $z_4 = -i$. Thus we have two real roots and one pair of complex conjugate roots. Therefore we have

$$p(z) = (z - 1)(z + 1)(z - i)(z + i) = (z - 1)(z + 1)(z^2 + 1).$$

Next let us look at $p_1(z) = z^8 - 2z^4 + 1$. If we note that $p_1(z) = (z^4 - 1)^2$, we can use the previous factorization to get

$$p_1(z) = (z - 1)^2 (z + 1)^2 (z - i)^2 (z + i)^2 = (z - 1)^2 (z + 1)^2 (z^2 + 1)^2.$$

Thus the roots are the same, but their multiplicities are different.

We now give a somewhat more complicated example. We let

$$p_2(z) = z^5 - 3\,z^4 + 8\,z^3 - 14\,z^2 + 16\,z - 8.$$

The roots of this polynomial are

$$z_1 = 1,\ z_2 = 1 + i,\ z_3 = 1 - i,\ z_4 = 2i,\ z_5 = -2i.$$

Thus there is one real root and two pairs of complex conjugate roots. The factorization in Proposition 6.2 becomes

$$p_2(z) = (z - 1)((z - 1)^2 + 1)(z^2 + 4) = (z - 1)(z^2 - 2z + 2)(z^2 + 4).$$

Remark 6.3. The result in Remark 4.3 can be generalized to an arbitrary polynomial. Again to simplify the statement we assume that the coefficient to the highest power is equal to one. Thus we consider a polynomial

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with roots (repeated with multiplicity) $z_1, z_2, \ldots, z_n$. These roots then satisfy

$$z_1 + z_2 + \cdots + z_n = -a_{n-1} \quad \text{and} \quad z_1 z_2 \cdots z_n = (-1)^n a_0. \tag{6.2}$$

References

[1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, Chapter 1 reprinted in Pearson Custom Publication 2011.