(Practice)Exam in Linear Algebra

First Year at The Faculties of Engineering and Science and of Health

This test has 9 pages and 15 problems. In two-sided print. It is allowed to use books, notes, photocopies etc. It is not allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

This exam set has two independent parts.

- Part I contains "essay problems". Here it is important that you explain the idea behind the solution, and that you provide relevant intermediate results.
- Part II is "multiple choice" problems. **The answers for Part II must be given on these sheets.**

Remember to write your full name (including middle names) together with your student number on each side of your answers. Number each page. Write the total number of pages and the page number on each page of the answers.

NAME: ______________________________

STUDENT NUMBER: ______________________________

Part I ("Essay-problems")

Problem 1 (8%).

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 0 & -4 \\ 2 & 2 & 0 & 5 \end{bmatrix}.$$

1. Find a basis for the column space of A.
2. Find a basis for the null space of A.
3. Find a basis for the row space of A.

Problem 2 (12%).

Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Find the eigenvalues of A.
2. Find a basis for each of the corresponding eigenspaces.
3. Specify whether A is diagonalisable. If so, find matrices P and D, such that D is diagonal, P is invertible, and $A = PDP^{-1}$.

Part II (Multiple-choice problems)

Problem 3 (5%)

Let R be the row reduced echelon form of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}$$

Specify the value of R_{24}:

☐ 0 ☐ $-\frac{1}{2}$ ☐ $\frac{1}{3}$ ☐ 2 ☐ $-\frac{3}{8}$

Problem 4 (10%).

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -5 & 7 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

Mark all correct statements below (notice: every incorrect mark cancels a correct one).

☐ A is not invertible.
☐ The linear transformation induced by A is injectiove (one-to-one).
☐ A is in row-echelon form.
☐ nullity $A = 1$.
☐ rank $A = 3$.
☐ nullity A + rank $A = 6$.
☐ The number 0 is an eigenvalue of A.
☐ A is in reduce row-echelon form.
☐ There is a vector $\mathbf{b} \in \mathbf{R}^4$, such that $A\mathbf{x} = \mathbf{b}$ is not consistent.
☐ $\det A = 0$.

Problem 5 (8%)

Given two 3×3-matrices A og B. Suppose that $\det A = -3$ and that B is an orthogonal matrix with $\det(B) > 0$. Answer the following questions:

a. Specify $\det B$:

☐ 0 ☐ -2 ☐ 1 ☐ 0.1

b. Specify $\det(AB)$:

☐ -2 ☐ 2 ☐ -3 ☐ 0

c. Specify $\det(-A)$:

☐ 1 ☐ -3 ☐ 1/2 ☐ 3

Problem 6 (7%).

Answer the following 4 true/false questions:

a. Let W be a subspace of $\mathbf{R}^6$ having dimension 4. Then $\dim(W^\perp) = 2$.

☐ True ☐ False

b. There exists a surjective (onto) linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$.

☐ True ☐ False

c. Suppose Q is a 4×4 ortogonal matrix. Then Q^5 is an ortogonal matrix.

☐ True ☐ False

d. A 3×3 matrix A with eigenvalues $1, 2$ and -3 is *both* invertible and diagonalizable.

☐ True ☐ False

Problem 7 (5%)

Which of the following statements are true (notice: every incorrect mark cancels a correct one):

☐ Any orthonormal set in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$, $n > 1$.
☐ The vectors in an orthonormal set in $\mathbf{R}^n$ are linearly independent.
☐ The vectors in an orthonormal set in $\mathbf{R}^n$ span $\mathbf{R}^n$.
☐ The number of vectors in an orthonormal set in $\mathbf{R}^n$ is at most n.

Problem 8 (5%)

Let C be given by

$$C = \begin{bmatrix} 2 & -3 & 0 & 3 \\ 0 & -2 & 2 & -3 \\ 0 & 3 & -2 & -3 \\ 0 & -1 & 1 & -2 \end{bmatrix}.$$

Then $\det(C)$ is:

☐ 2 ☐ -3 ☐ 0 ☐ 4 ☐ -2/5

Problem 9 (5%)

Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \\ -1 & 1 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$

Answer the following 2 true/false questions:

i. The vector $\mathbf{b}$ is contained in $\mathrm{Col}(A)$.

☐ True ☐ False

ii. The vector $\mathbf{b}$ is contained in $\mathrm{Nul}(A)$.

☐ True ☐ False

Problem 10 (5%)

The following basis is given

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

for $\mathbf{R}^3$. Denote $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and consider the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Answer the following two questions:

i. $\mathcal{B}$ is an orthonormal basis $\mathbf{R}^3$.

☐ True ☐ False

ii. The third coordinate of $[\mathbf{v}]_{\mathcal{B}}$ is given by:

☐ $-\sqrt{2}$ ☐ 3 ☐ $\frac{1}{2}$ ☐ $-\frac{1}{\sqrt{3}}$ ☐ $\frac{1}{\sqrt{2}}$

Problem 11 (8%)

The row-echelon reduced form of the matrix

$$A = \begin{bmatrix} -2 & 2 & 3 & 1 & -3 & -2 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ -1 & 0 & 1 & -3 & -2 & 2 \end{bmatrix}$$

is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 10 & 2 & -7 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 7 & 0 & -5 \end{bmatrix}.$$

Answer the following 4 questions about A:

a. The number of pivots of A is:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

b. The number of free variables in the system of equations $A\mathbf{x} = \mathbf{0}$ is:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

c. Let T be the linear transformation $T : \mathbf{R}^6 \to \mathbf{R}^d$ given by $T(\mathbf{x}) = A\mathbf{x}$. The number d is:

☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

d. The linear transformation $T(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^6$, is surjective (onto).

☐ True ☐ False

Problem 12 (5%)

Consider the system of equations

$$\begin{cases} x_1 \phantom{{}-x_2} + x_3 = 3 \\ x_1 - x_2 - x_3 = 1 \\ -x_1 + x_2 \phantom{{}-x_3} = 4 \end{cases}$$

This system has (mark only one statement):

- ☐ No solution
- ☐ An infinite number of solutions
- ☐ A uniquely determined solution
- ☐ None of the above statements apply.

Problem 13 (5%)

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

Which of the following statements hold true (mark only one statement):

- ☐ A's columns are linearly dependent
- ☐ $\det(A) = 1$
- ☐ A is not invertible
- ☐ None of the above statements apply.

Problem 14 (7%)

The number of linearly independent eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

is given by:

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

Problem 15 (5%)

Consider the matrix product AB, where

$$A = \begin{bmatrix} -2 & 0 & -3 & 2 & -2 & -3 \\ 1 & -2 & -2 & -2 & 3 & -3 \\ 2 & 2 & 1 & 0 & -3 & 1 \\ 3 & 3 & -1 & 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & -2 \\ -3 & -1 & -3 & -3 \\ -1 & 3 & -1 & -2 \\ 1 & 3 & 2 & -2 \\ -3 & -2 & -3 & 0 \\ -2 & -2 & -2 & -3 \end{bmatrix}.$$

The value of entry $(2, 4)$ in AB, i.e. $(AB)_{24}$, is:

☐ 2 ☐ -12 ☐ 21 ☐ -22 ☐ -13/12.