Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

26 February 2020, 9:00–13:00

This test consists of 11 numbered pages with 13 problems. For each problem there is given a number of points. The total for all problems is 100 points.

Allowed aids: Books, notes, xerox copies and print.

Not allowed: Electronic aids such as calculator or mathematical applications on the computer. Electronic documents.

Facit

Full points are given if all the correct and no wrong answers are checked. A wrong answer cancels a correct answer in the same question.

Problem 1 (10 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix multiplication results in the matrix $C = AB$.

(a) (5 points). What is the size of the matrix C?

☑ 2×3 ☐ 3×3 ☐ 2×4

☐ 3×2 ☐ 4×3 ☐ 4×2

(b) (5 points). What is the entry c_{13}?

☐ -3 ☐ 0 ☐ 4

☑ -1 ☐ 1 ☐ 3

Problem 2 (2 points)

Three matrices are given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & 4 \\ -1 & -1 & 5 \end{bmatrix}.$$

Which matrix products are defined?

☐ BA and BC ☐ AC, BC and BA ☐ C^2, A^2 and B^2

☑ CA, CB and BC ☐ C^TA, CA and AC ☐ AB^T, CB and CA

Problem 3 (10 points)

Consider the system of equations

$$\begin{aligned} x_1 - 3x_2 &= 5 \\ -x_1 + x_2 + 5x_3 &= 2 \\ -x_2 - x_3 &= 0 \end{aligned}$$

Mark the two correct statements below:

- ☐ The system of equations has no solution.
- ☐ $x_1 = 1$, $x_2 = 4$ and $x_3 = 1$ is a solution to the system of equations.
- ☑ $x_1 = 2$, $x_2 = -1$ and $x_3 = 1$ is a solution to the system of equations.
- ☐ The system of equations has exactly two solutions.
- ☐ The system of equations has infinitely many solutions.
- ☑ The system of equations has exactly one solution..

Problem 4 (10 points)

The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 6 & 6 & 6 \\ 3 & -6 & 3 \\ 2 & -21 & 2 \end{bmatrix}$$

is $-t^3 + 2t^2 + 3t$.

(a) (2 points). Which three of the following are eigenvalues of A?

☐ -3 ☐ -2 ☑ -1 ☑ 0 ☐ 1 ☐ 2 ☑ 3

(b) (2 points). Which one of the following is an eigenvector of A?

☐ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ☑ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(c) (2 points). Is A invertible?

☐ Yes ☑ No

(d) (2 points). Is A diagonalisable?

☑ Yes ☐ No

(e) (2 points). If $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then EA equals:

☐ $\begin{bmatrix} 6 & 6 & 6 \\ 3 & -6 & 3 \\ 9 & 0 & 9 \end{bmatrix}$ ☐ $\begin{bmatrix} 3 & 0 & 3 \\ 3 & -6 & 3 \\ 2 & -21 & 2 \end{bmatrix}$

☐ $\begin{bmatrix} 6 & 6 & 6 \\ -9 & 0 & -9 \\ 2 & -21 & 2 \end{bmatrix}$ ☑ $\begin{bmatrix} 6 & 6 & 6 \\ 9 & 0 & 9 \\ 2 & -21 & 2 \end{bmatrix}$

Problem 5 (10 points)

Let $T\colon \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

(a) (2 points). What is n?

☐ 0 ☐ 1 ☐ 2 ☑ 3 ☐ 4

(b) (2 points). What is m?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☑ 4

(c) (2 points). What is the rank Rank(A)?

☐ 0 ☐ 2 ☐ 4 ☐ 6
☐ 1 ☑ 3 ☐ 5

(d) (2 points). What is the nullity Nullity(A)?

☑ 0 ☐ 2 ☐ 4 ☐ 6
☐ 1 ☐ 3 ☐ 5

(e) (2 points). For the sytem of equations $Ax = b$ the following holds:

- ☐ It is always consistent.
- ☑ It has either none or exactly one solution.
- ☐ It has either none or infinitely many solutions.
- ☐ None of the previous statements holds.

Problem 6 (10 points)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) (5 points) What is $A\mathbf{v}$?

☐ $\begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ☐ None of these.

(b) (5 points) What is the inverse of A?

☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 7 (9 points)

Let $\mathbf{v}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{u} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$.

(a) (3 points). Are $\mathbf{v}_1$ and $\mathbf{v}_2$ orthandonal?

☑ Yes ☐ No ☐ Neither yes nor no

(b) (3 points). What is the orthogonal projection of $\mathbf{u}$ onto W?

☐ $\begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ ☑ $\begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$ ☐ $\begin{bmatrix} 9 \\ -1 \\ 6 \end{bmatrix}$

(c) (3 points). What is the distance from $\mathbf{u}$ to W?

☐ 2 ☐ $\frac{1}{2}$ ☐ $\frac{1}{6}$

☑ 0 ☐ $\frac{3}{4}$ ☐ None of these.

Problem 8 (2 points)

Let $T\colon \mathcal{R}^2 \to \mathcal{R}^4$ and $S\colon \mathcal{R}^4 \to \mathcal{R}^2$ be linear transformations and define $U = TS\colon \mathcal{R}^4 \to \mathcal{R}^4$, i.e., U is the linear transformation, where you first apply S and then T, and $V = ST\colon \mathcal{R}^2 \to \mathcal{R}^2$, i.e., V is the transformation, where you first apply T and then S. Which four of the following statements can you conclude without more knowledge about T, S, U and V?

- [x] T cannot be surjective (onto)
- [ ] T cannot be injective (1-1)
- [ ] S cannot be surjective (onto)
- [x] S cannot be injective (1-1)
- [x] U cannot be surjective (onto)
- [x] U cannot be injective (1-1)
- [ ] V cannot be surjective (onto)
- [ ] V cannot be injective (1-1)

Problem 9 (6 points)

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 4 \\ -3 \\ 16 \end{bmatrix}$. If, by applying the Gram-Schmidt process on $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, one gets orthogonal vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, then:

(a) (2 points)

- [ ] $\mathbf{v}_1 = -\mathbf{u}_1$
- [x] $\mathbf{v}_1 = \mathbf{u}_1$
- [ ] None of these.

(b) (2 points)

- [ ] $\mathbf{v}_2 = \mathbf{u}_2 - \mathbf{v}_1$
- [x] $\mathbf{v}_2 = \mathbf{u}_2 - 2\mathbf{v}_1$
- [ ] None of these.
- [ ] $\mathbf{v}_2 = \mathbf{u}_2 + \mathbf{b}_1$
- [ ] $\mathbf{v}_2 = \mathbf{u}_2 + 2\mathbf{v}_1$

(c) (2 points)

- [ ] $\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{v}_1 + \mathbf{v}_2$
- [ ] $\mathbf{v}_3 = \mathbf{u}_2 - 2\mathbf{v}_1$
- [ ] None of these.
- [ ] $\mathbf{v}_3 = \mathbf{u}_3 - 2\mathbf{v}_1 - \mathbf{v}_2$
- [x] $\mathbf{v}_3 = \mathbf{u}_3$

Problem 10 (10 points)

Let A and B be 3×3-matrices with determinants $\det(A) = -4$ and $\det(B) = 0$.

(a) (2 points). What is $\det(A^T)$?

☐ 2 ☐ 0 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

(b) (2 points). What is $\det(-A)$?

☐ 2 ☐ 0 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

(c) (2 points). What is $\det(AB^{-1})$?

☐ 2 ☐ 0 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

(d) (2 points). What is $\det(-B^2)$?

☐ 2 ☐ 0 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

(e) (2 points). What is $\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 1 & 3 & 2 \end{bmatrix}\right)$?

☐ 2 ☐ 0 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

Problem 11 (10 points)

The matrix A is row reduced to the matrix R, where

$$R = \begin{bmatrix} 1 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 7 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5\ \mathbf{a}_6\ \mathbf{a}_7]$ where $\mathbf{a}_i$ is the i'th column of A.

(a) (2 points). What is the nullity Nullity(A)?

- ☐ 0
- ☐ 1
- ☑ 2
- ☐ 3
- ☐ 4
- ☐ 5
- ☐ 6
- ☐ 7
- ☐ 8
- ☐ That can't be determined with the given information.

(b) (2 points). What is the rank Rank(A)?

- ☐ 0
- ☐ 1
- ☐ 2
- ☐ 3
- ☐ 4
- ☑ 5
- ☐ 6
- ☐ 7
- ☐ 8
- ☐ That can't be determined with the given information.

(c) (2 points). Which columns in A are pivot columns?

- ☐ $\mathbf{a}_4$ and $\mathbf{a}_6$
- ☐ $\mathbf{a}_2$, $\mathbf{a}_4$ and $\mathbf{a}_6$
- ☐ $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$
- ☑ $\mathbf{a}_1$,$\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_5$, $\mathbf{a}_7$
- ☐ All of them
- ☐ None of them

(d) (2 points). Which of the following statements is correct?

- ☐ $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ are linearly independent.
- ☐ $\mathbf{a}_4 = 3\mathbf{a}_1 - 3\mathbf{a}_3$
- ☑ $\mathbf{a}_4 = -3\mathbf{a}_1 + 3\mathbf{a}_2 + \mathbf{a}_3$
- ☐ $\mathbf{a}_4 = 2\mathbf{a}_2 + 3\mathbf{a}_3$
- ☐ $\mathbf{a}_1 = 2\mathbf{a}_2 - 3\mathbf{a}_3$
- ☐ $\mathbf{a}_5$ is a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

(e) (2 points). If $A = [B\ \mathbf{b}]$ is the augmented matrix of a system of linear equations in $x_1, x_2, \ldots, x_5$. Which of the following statements are then correct?

- ☑ The system of equations has no solutions.
- ☐ The system of equations has infinitely many solutions.
- ☐ The system of equations has exactly one solution.
- ☐ That can't be determined with the given information.

Problem 12 (8 points)

A and B are two quadratic matrices, $\mathbf{v}$ is an eigenvector of A with eigenvalue 8, $\mathbf{w} = A\mathbf{v}$ is an eigenvector of B with eigenvalue -4.B is invertible.

(a) (2 points). Is $\mathbf{v}$ an eigenvector for AB?

- ☐ Yes, with eigenvalue -2
- ☐ Yes, with eigenvalue 4
- ☐ Yes, with eigenvalue -8
- ☐ Yes, with eigenvalue 64
- ☑ Yes, with eigenvalue -32
- ☐ No

(b) (2 points). Is $\mathbf{v}$ an eigenvector for A^2?

- ☐ Yes, with eigenvalue -2
- ☐ Yes, with eigenvalue 4
- ☐ Yes, with eigenvalue -8
- ☑ Yes, with eigenvalue 64
- ☐ Yes, with eigenvalue -32
- ☐ No

(c) (2 points). Is $\mathbf{v}$ an eigenvector for $4B - 2A$?

- ☐ Yes, with eigenvalue -2
- ☐ Yes, with eigenvalue 4
- ☐ Yes, with eigenvalue -8
- ☐ Yes, with eigenvalue 64
- ☑ Yes, with eigenvalue -32
- ☐ No

(d) (2 points). Is $\mathbf{v}$ an eigenvector for AB^{-1}?

- ☑ Yes, with eigenvalue -2
- ☐ Yes, with eigenvalue 4
- ☐ Yes, with eigenvalue -8
- ☐ Yes, with eigenvalue 64
- ☐ Yes, with eigenvalue -32
- ☐ No

Problem 13 (3 points)

In MATLAB's Command Window the following is typed:

```
>> u = [1; 1; 1; 1];
>> v = [1; 2; 3; 4];
>> w = [1; 3; 4; 10];
>> z = [1; 4; 10; 1];
>> A = [u v w z];
>> B = [u v w z w-u];
>> rref(A)
 ans =
         1    0    0   -5
         0    1    0    9
         0    0    1   -3
         0    0    0    0
```

What do you get if you then type `rref(B)` in MATLAB's Command Window?

☐
```
ans =
         1    0    0   -5    0
         0    1    0    9    0
         0    0    1   -3    0
         0    0    0    0    1
```

☐
```
ans =
         1    0    0   -5   -1    0
         0    1    0    9    0    0
         0    0    1   -3    0    1
         0    0    0    0    0    0
```

☐
```
ans =
         1    0    0    1    1
         0    1    0    3    0
         0    0    1    4   -1
         0    0    0    0    0
```

☐
```
ans =
         1    0    0    1    0   -1
         0    1    0    3    0    0
         0    0    1    4    1    0
         0    0    0    0    0    0
```

☑
```
ans =
         1    0    0   -5   -1
         0    1    0    9    0
         0    0    1   -3    1
         0    0    0    0    0
```

☐
```
ans =
         1    0    0    1   -5    0
         0    1    0    3    9    0
         0    0    1    4   -3    0
         0    0    0    0    0    1
```