For at finde den danske version af prøven, begynd i den modsatte ende!

Please disregard the Danish version on the back if you participate in this English version of the exam.

Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

August 22, 2017, 9:00 – 13:00

This test consists of 9 pages and 14 problems. All problems are "multiple choice" problems. Your answers must be given by marking the relevant boxes on these sheets.

It is allowed to use books, notes, xerox copies etc. It is **not allowed** to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.
Problems 3.2, 4, 8, 11, 12 and 13 can have more than one correct solution.
These problems will be evaluated as follows: Every wrong mark will annul one correct mark.

Remember to fill in your full name (including middle names) together with your student number below.
Moreover, please mark the team that you participate in.

Good luck!

NAME:______________________________

STUDENT NUMMBER:______________________________

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with solutions

Problem 1 (5 points)

1. Mark the determinant of the matrix $A = \begin{bmatrix} 5 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

   ☐ 15 ☐ 13 ☑ 11 ☐ -13 ☐ -7

2. A 3×3 matrix B has determinant -2.
   Which of the following numbers agrees with the determinant of the inverse matrix B^{-1}?

   ☐ -2 ☐ $\frac{1}{8}$ ☐ 2
   ☑ $-\frac{1}{2}$ ☐ $-\frac{1}{8}$ ☐ none of them

Problem 2 (8 points)

1. Is the system of equations

$$\begin{array}{rrrcl} x_1 & +3x_2 & -x_3 & = & 1 \\ 2x_1 & +x_2 & & = & 0 \\ x_1 & +8x_2 & -3x_3 & = & 3 \end{array}$$

   consistent?

   ☑ Yes ☐ No

2. How many solutions has this system of equations?

   ☐ 1 ☐ none
   ☐ 2 ☑ infinitely many

3. Is the system of equations

$$\begin{array}{rrrcl} x_1 & +3x_2 & -x_3 & = & 0 \\ 2x_1 & +x_2 & & = & 1 \\ x_1 & +8x_2 & -3x_3 & = & 3 \end{array}$$

   consistent?

   ☐ Yes ☑ No

4. How many solutions has this system of equations?

   ☐ 1 ☑ none
   ☐ 4 ☐ infinitely many

Problem 3 (11 points)

This problem concerns the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

1. Which of the following polynomials is the charakteristic polynomial of A?

   - [ ] $2\lambda^2 - 8\lambda + 6$
   - [ ] $-\lambda^3 + 4\lambda^2 - 11\lambda + 4$
   - [x] $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$
   - [ ] none of these

2. Which of the following vectors are eigenvectors of the matrix A?

   - [ ] $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
   - [x] $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
   - [x] $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$
   - [ ] $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
   - [x] $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
   - [ ] $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3. Is the matrix A diagonalisable?

   - [x] Yes
   - [ ] No

4. Which of the following numbers conincides with $\det(A)$?

   - [ ] 0
   - [ ] 2
   - [ ] 4
   - [x] 6
   - [ ] 8
   - [ ] -6

5. Is the matrix A regular/invertible?

   - [x] Yes
   - [ ] No

Problem 4 (6 points)

Departing from the matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ the matrix products $C = AB$ and $D = BA$ can be determined.

Mark the true ones among the following assertions concerning the coefficients c_{ij} of C, resp. d_{ij} of D:

- [x] $c_{11} = d_{11}$
- [ ] $c_{13} = d_{13}$
- [ ] $c_{23} = d_{23}$
- [ ] $c_{12} = d_{12}$
- [x] $c_{22} = d_{22}$
- [x] $c_{33} = d_{33}$

Problem 5 (9 points)

A 3×3-matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ has three mutually orthogonal column vectors $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ -1 \\ a_{32} \end{bmatrix}$ og $\mathbf{a}_3 = \begin{bmatrix} -1 \\ a_{23} \\ a_{33} \end{bmatrix}$.

1. What are the correct values of a_{32}, a_{23} og a_{33}?

   - ☐ $a_{32} = a_{23} = a_{33} = -2$
   - ☐ $a_{32} = 1, a_{23} = 2, a_{33} = -2$
   - ☑ $a_{32} = a_{33} = -2, a_{23} = 2$
   - ☐ $a_{32} = a_{33} = -2, a_{23} = 1$

2. Which of the following matrices coincides with AA^T?

   - ☐ $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$
   - ☐ $\begin{bmatrix} 9 & 0 & 9 \\ 0 & 9 & 0 \\ 9 & 0 & 9 \end{bmatrix}$
   - ☐ $\begin{bmatrix} 9 & 0 & 6 \\ 0 & 9 & 0 \\ 6 & 0 & 9 \end{bmatrix}$
   - ☑ $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

3. What is the determinant $\det(A)$?

   - ☐ 1
   - ☐ 9
   - ☑ 27
   - ☐ 729

Problem 6 (8 points)

A line l in the plane is given by the equation $3x_1 + 4x_2 = 0$; furthermore consider the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 25 \end{bmatrix}$.

1. Which of the following vectors is the orthogonal projection of the vector $\mathbf{v}$ on the line l?

   - ☐ $\begin{bmatrix} 12 \\ 16 \end{bmatrix}$
   - ☑ $\begin{bmatrix} -12 \\ 9 \end{bmatrix}$
   - ☐ $\begin{bmatrix} -300 \\ 225 \end{bmatrix}$
   - ☐ $\begin{bmatrix} 0 \\ \frac{25}{3} \end{bmatrix}$

2. Which of the following numbers coincides with the distance from the point $P : (0, 25)$ to the line l?

   - ☑ 20
   - ☐ 15
   - ☐ $\frac{50}{3}$
   - ☐ $100\sqrt{13}$

Problem 7 (8 points)

A counter-clockwise rotation $T : \mathcal{R}^2 \to \mathcal{R}^2$ in the plane around the origin by an angle $\theta = \frac{\pi}{4}$ has a standard matrix A.

1. Which of the following matrices coincides with A?

   ☐ $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ ☑ $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

2. Is the matrix A regular/invertible?

   ☑ Yes ☐ No

3. Is the matrix A diagonalisable?

   ☐ Yes ☑ No

4. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ og $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form an ordered basis $\mathcal{B}$ for $\mathcal{R}^2$.
   Which of the following matrices is the matrix $T_{\mathcal{B}}$ that describes the rotation T with respect to the ordered basis $\mathcal{B}$?

   ☐ $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ ☑ $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Problem 8 (8 points)

This problem concerns the following vectors in 3-space:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{R}^3.$$

1. Which of the following sets of vectors span $\mathcal{R}^3$?

   ☐ $\mathbf{v}_1, \mathbf{v}_2$ ☑ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ☑ $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$

   ☐ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_7$ ☑ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_6$ ☑ $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$

2. Which of the following sets of vectors are linearly independent?

   ☑ $\mathbf{v}_1$ ☐ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_7$ ☐ $\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$

   ☑ $\mathbf{v}_1, \mathbf{v}_2$ ☑ $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ ☑ $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5$

Problem 9 (6 points)

This problem concerns the following system of equations.

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & + & 2x_2 & & & = & 1 \\ x_1 & + & x_2 & + & 3x_3 & = & -2 \end{array}.$$

1. Which of the following matrices is the augmented matrix $[A\ \mathbf{b}]$ that represents the system of equations?

   ☐ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & -2 \end{bmatrix}$

   ☐ $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & -2 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

2. Which of the following matrices is the reduced echelon matrix that is row-equivalent to $[A\ \mathbf{b}]$?

   ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

   ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

3. Which of the following assertions are true?

   ☐ The system is inconsistent

   ☐ $x_1 = 7, x_2 = -3, x_3 = -2$ is one of several solutions of the system.

   ☑ $x_1 = 7, x_2 = -3, x_3 = -2$ is the only solution of the system.

   ☐ $x_1 = 6, x_2 = -2, x_2 = -2$ is one of several solutions of the system.

   ☐ $x_1 = 6, x_2 = -2, x_3 = -2$ is the only solution of the system.

Problem 10 (5 points)

This problem concerns the three elementary matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ and the matrix } A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

Mark the matrix E_i, $1 \le i \le 3$, for which

1. $E_i A = \begin{bmatrix} 0 & 1 & 2 \\ 6 & 8 & 10 \\ 6 & 7 & 8 \end{bmatrix}$;

   ☐ E_1 ☑ E_2 ☐ E_3

2. $E_i A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 5 & 4 \end{bmatrix}$;

   ☐ E_1 ☐ E_2 ☑ E_3

3. $E_i A = \begin{bmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$.

   ☑ E_1 ☐ E_2 ☐ E_3

4. Is every elementary matrix invertible?

   ☑ Yes ☐ No

5. Is the product of two elementary matrices always an elementary matrix?

   ☐ Yes ☑ No

Problem 11 (8 points)

This problem is about a 2×2 matrix A. Mark the correct ones in the list of assertions below:

☑ If $\det(A)$ is an integer, then $\det(A^T)$ is also an integer.

☐ If $\det(A)$ is an integer with $\det A \neq 0$, then $\det(A^{-1})$ is also an integer.

☑ If A is a rotation matrix, then $\det(A) = 1$.

☐ If A is a reflection matrix, then $\det(A) = 1$.

Problem 12 (6 points)

This problem concerns the matrix $B = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix}$ and the vectors $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{R}^2$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \in \mathcal{R}^3$.

Mark the correct assertions in the list below:

- [x] $\mathbf{b}$ is contained in the column space Col B.
- [ ] $\mathbf{c}$ is contained in the column space Col B.
- [ ] $\mathbf{b}$ is contained in the null space Null B.
- [x] $\mathbf{c}$ is contained in the null space Null B.
- [x] The column space Col B coincides with $\mathcal{R}^2$.
- [ ] The null space Null B coincides with $\mathcal{R}^3$.

Problem 13 (6 points)

The following commands are entered into MATLAB's Command Window:

```
>> a = [1; 1; 1; 1];
>> b = [1; 2; 1; 1];
>> c = [1; 0; 3; -4];
>> d = [2; 1; -2; 5];
>> e = [6; -2; 12; -16];
>> C = [a b c d e];
>> rref(C);

ans =

 1   0   0   0    1
 0   1   0   0   -2
 0   0   1   0    5
 0   0   0   1    1
```

Mark the correct ones among the following assertions:

- [ ] `e` is a row vector.
- [x] `e` is a column vector.
- [x] `C` is a 4×5 matrix.
- [ ] `C` is a 5×4 matrix.
- [ ] C's nullity (the dimension of Null C) can be calculated by entering
`>> 4 - rank (C);`
- [x] C's nullity (the dimension of Null C) can be calculated by entering
`>> 5 - rank (C);`

Opgave 14 (6 point)

Consider an augmented matrix

$$[A\ \mathbf{b}] = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 4 & -3 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

1. Which of the following systems of equations corresponds to the equation $A\mathbf{x} = \mathbf{b}$?

   ☑ $$\begin{array}{rcrcrcrcrcl} -x_1 & & & + & x_3 & + & x_4 & & & = & 0 \\ x_1 & + & 4x_2 & - & 3x_3 & & & + & x_5 & = & 0 \\ x_1 & - & 2x_2 & + & x_3 & & & & & = & 1 \end{array}$$

   ☐ $$\begin{array}{rclcrcrcrcr} -1 & = & & & x_2 & + & x_3 & & & & \\ 1 & = & 4x_1 & - & 3x_2 & & & + & x_4 & & \\ 1 & = & -2x_1 & + & x_2 & & & & & + & x_5 \end{array}$$

   ☐ $$\begin{array}{rcrcrcrcrcrcl} -x_1 & & & + & x_2 & + & x_4 & & & & & = & 0 \\ x_1 & + & 4x_2 & - & 3x_3 & & & + & x_5 & & & = & 0 \\ x_1 & - & 2x_2 & + & x_3 & & & & & + & x_6 & = & 0 \end{array}$$

   Reducing $[A\ \mathbf{b}]$ to reduced echelon form results in the matrix

   $$[H\ \mathbf{c}] = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

2. Which of the following is the rank of the coefficient matrix A?

   ☐ 1 ☐ 2 ☑ 3 ☐ 4 ☐ 5

3. Which of the following is the rank of total matrix $[A\ \mathbf{b}]$?

   ☐ 1 ☐ 2 ☑ 3 ☐ 4 ☐ 5 ☐ 6

4. Which of the following is the nullity of the matrix A?

   ☐ 0 ☐ 1 ☑ 2 ☐ 3 ☐ 4 ☐ 5

5. Does $x_1 = 0, x_2 = -1, x_3 = -1, x_4 = 1, x_5 = 1$
   solve the system corresponding to the equation $A\mathbf{x} = \mathbf{b}$?

   ☑ Yes ☐ No

6. Is $x_1 = 0, x_2 = -1, x_3 = -1, x_4 = 1, x_5 = 1$
   the only solution of the system corresponding to the equation $A\mathbf{x} = \mathbf{b}$?

   ☐ Yes ☑ No