Exam in Linear Algebra

First Year at The Faculty of Engineering and Science
and The Technical Faculty of IT and Design

February 17, 2017, 9.00-13.00

This test has 9 pages and 14 problems. All the problems are "multiple choice" problems. **The answers must be given on these sheets.**

It is allowed to use books, notes, photocopies etc. It is **not** allowed to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.

Remember to write your full name (including middle names) together with your student number below.

NAME: _____

STUDENT NUMBER: _____

In all problems: *there is only one correct answer to each question.*

Problem 1 (9%)

Consider the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and answer the following questions.

1. Is A in row echelon form?

   ☐ Yes ☒ No

2. Is B in row echelon form?

   ☒ Yes ☐ No

3. Is A in reduced row echelon form?

   ☐ Yes ☒ No

4. Is B in reduced row echelon form?

   ☐ Yes ☒ No

5. Can B be obtained from A by elementary row operations?

   ☐ Yes ☒ No

6. Can A be obtained from B by elementary row operations?

   ☐ Yes ☒ No

Problem 2 (4 %)

Let $A = [\mathbf{a}_1\ \mathbf{a}_2]$ be a matrix with 4 rows and let $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4\ \mathbf{b}_5]$ be such that $C = AB$ is defined.

1. How many rows are there in the matrix B?

   ☒ 2 ☐ 3 ☐ 4 ☐ 5

2. How many rows are there in the matrix C?

   ☐ 2 ☐ 3 ☒ 4 ☐ 5

Problem 3 (10 %)

Let $A = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$ and let $B = \frac{1}{5}A$.

1. Answer the following true/false problems.

A is an orthogonal matrix.	☐ True	☒ False
B is an orthogonal matrix.	☒ True	☐ False
A is a symmetric matrix.	☒ True	☐ False
$B^{-1} = -B$	☐ True	☒ False
$B^{-1} = B$	☒ True	☐ False

2. What is the determinant of B?

   ☐ -5 ☒ -1 ☐ 0 ☐ 2 ☐ 5

3. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigenvector of B with eigenvalue 1. What is the value of x_1 if $x_2 = 1$.

☐ -3 ☐ -1 ☐ 0 ☐ 1 ☒ 3

Problem 4 (10 %)

Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. The augmented matrix $\begin{bmatrix} 1 & -1 & 2 & 1 & 0 \\ 2 & 1 & 4 & -1 & 3 \\ 1 & 1 & 2 & 1 & 4 \end{bmatrix}$ has the following reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

1. Answer the following problems about pivot columns of A:

   column 1 is a pivot column. ☒ True ☐ False

   column 2 is a pivot column. ☒ True ☐ False

   column 3 is a pivot column. ☐ True ☒ False

   column 4 is a pivot column. ☒ True ☐ False

2. What is the rank of A?

   ☐ 0 ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5

3. What is the nullity of A?

   ☐ 0 ☒ 1 ☐ 2 ☐ 3 ☐ 4

4. Let $\mathbf{x}$ be a solution of $A\mathbf{x} = \mathbf{b}$. What is x_2 ?

   ☐ 1
   ☒ 2
   ☐ 3
   ☐ x_2 is a free variabel.

Problem 5 (4 %)

Let $A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 5 & -3 \\ 1 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -2 & 1 \\ 5 & 2 & 2 \end{bmatrix}$.
Let $C = AB$. What is the number c_{13} ?

☐ -7 ☒ -5 ☐ -3 ☐ 8 ☐ 11

Problem 6 (10 %)

Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ and let $W = \text{Span}\{\mathbf{v}\}$. Let $\mathbf{u} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ and let $\mathbf{w}$ be the orthogonal projection of $\mathbf{u}$ on W.

1. What is the third component of $\mathbf{w}$ (i.e. w_3)?

   ☐ -4 ☐ -2 ☒ -1 ☐ 0 ☐ 1 ☐ 4

2. Let $\mathbf{z}$ be the orthogonal projection of $\mathbf{u}$ on $W^{\perp}$. What is the third component of $\mathbf{z}$ (i.e. z_3)?

   ☐ -2 ☐ -1 ☐ 2 ☒ 3 ☐ 4 ☐ 6

3. What is the dimension of $W^{\perp}$?

   ☐ 0 ☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5

Problem 7 (6 %)

Let A and B be 3×3 matrices with $\det A = 5$ and $\det B = 3$.

1. What is $\det(-2A)$?

   ☐ -200 ☐ -50 ☒ -40 ☐ -10 ☐ 5 ☐ 40

2. What is $\det AB^T$?

   ☐ 2 ☐ 3 ☐ 5 ☒ 15 ☐ $\frac{3}{5}$ ☐ $\frac{5}{3}$

3. What is $\det AB^{-1}$?

   ☐ 1 ☐ $\frac{1}{15}$ ☐ $\frac{3}{5}$ ☒ $\frac{5}{3}$ ☐ 2 ☐ 15

Problem 8 (10 %)

The characteristic polynomial of $A = \begin{bmatrix} -8 & 4 & -2 & 10 \\ 0 & -1 & 0 & 0 \\ 2 & -2 & 0 & -2 \\ -7 & 4 & -2 & 9 \end{bmatrix}$ is $t(t-2)(t+1)^2$.

1. Which one of the following is an eigenvalue of A?

   ☐ -2 ☒ -1 ☐ 1 ☐ 4

2. Which one of the following is an eigenvector of A?

   ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ☒ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

3. Is A invertible?

   ☐ Yes ☒ No

Problem 9 (4 %)

Let

$$A = \begin{bmatrix} 2 & 4 & 1 & 2 \\ 2 & 4 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

What is the determinant of A?

☐ 12 ☐ 10 ☒ 5 ☐ 0 ☐ -5 ☐ -10 ☐ -12

Problem 10 (4 %)

What is the number of solutions of the following system of linear equations

$$\begin{aligned} x_1 - x_2 \quad + x_4 &= 0 \\ x_1 - x_2 + x_3 + x_4 &= 0 \\ -x_1 + x_2 \quad - x_4 &= 1 \end{aligned}$$

☒ 0
☐ 1
☐ infinitely many.

Problem 11 (6%)

Let T be the linear transformation with standard matrix $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$.

$\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \}$ is a basis for $\mathcal{R}^2$.

Which one of the following is the matrix representation of T with respect to $\mathcal{B}$, denoted by $[T]_{\mathcal{B}}$?

☐ $\begin{bmatrix} -18 & -7 \\ 4 & 11 \end{bmatrix}$ ☒ $\begin{bmatrix} 11 & -18 \\ 4 & -7 \end{bmatrix}$ ☐ $\begin{bmatrix} 4 & 11 \\ -7 & -18 \end{bmatrix}$ ☐ $\begin{bmatrix} -7 & 4 \\ 11 & -18 \end{bmatrix}$

Problem 12 (10%)

Let $T : \mathcal{R}^n \to \mathcal{R}^m$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

1. What is the value of n ?

   ☐ 2 ☒ 3 ☐ 4 ☐ 5 ☐ 6

2. What is the value of m ?

   ☐ 2 ☐ 3 ☐ 4 ☒ 5 ☐ 6

3. What the rank of A ?

   ☐ 2 ☒ 3 ☐ 4 ☐ 5

4. What is the dimension of the null space of T ?

   ☒ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

5. Is T one-to-one?

   ☒ Yes ☐ No

6. Is T onto?

   ☐ Yes ☒ No

Problem 13 (6%)

Let $A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ and let $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

1. Is $\mathbf{b}$ contained in Col A? ☒ Yes ☐ No
2. Is $\mathbf{c}$ contained in Col A? ☐ Yes ☒ No
3. Is $\mathbf{b}$ contained in Null A? ☐ Yes ☒ No
4. Is $\mathbf{c}$ contained in Null A? ☒ Yes ☐ No

Problem 14 (7%)

The following commands are entered in the MATLAB Command Window:

```
>> A = [1 1 1 1; 1 2 3 4; 0 1 0 1; 0 0 0 1];
>> b = [1; 0; 1; 2];
>> T = [A b];
```

1. What is the size of the matrix T ?

   ☐ 1×20 ☐ 2×16 ☐ 4×4 ☒ 4×5 ☐ 5×4

2. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$. Which one of the following combinations of MATLAB commands computes $\mathbf{x}$?

   ☐
   ```
   >> R = rref(A); x = R(:,4)
   ```

   ☐
   ```
   >> R = rref(A); x = R(5,:)
   ```

   ☐
   ```
   >> R = rref(T); x = R(:,4)
   ```

   ☒
   ```
   >> R = rref(T); x = R(:,5)
   ```

   ☐
   ```
   >> R = rref(T); x = R(5,:)
   ```