Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

15 June 2020 9:00-13:00

Full marks are given if all the correct and no wrong answers are checked. A wrong answer cancels a correct answer in the same question.

At the exam, Moodle chose one of the problems 1A and 1B at random for each student.

Problem 1A (6 points)

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + x_3 &= 1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

(a) Mark the true statement(s):

- ☐ The system has no solutions
- ☐ $x_1 = x_2 = 1$ and $x_3 = -1$ is a solution to the system
- ☐ $x_1 = -1$, $x_2 = 0$ and $x_3 = 3$ is a solution to the system
- ☐ The system has exactly two solutions
- ☐ The system has infinitely many solutions
- ☐ The system has exactly one solution

Problem 1B (6 points)

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + x_3 &= 1 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

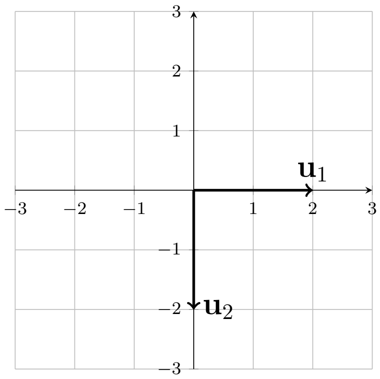
(a) Mark the true statement(s):

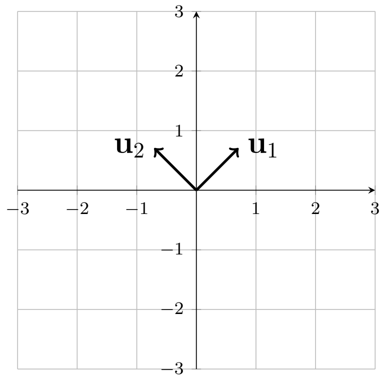
- ☐ The system has no solutions
- ☐ $x_1 = x_2 = 1$ and $x_3 = -1$ is a solution to the system
- ☐ $x_1 = -1$, $x_2 = 0$ and $x_3 = 3$ is a solution to the system
- ☐ The system has exactly two solutions
- ☐ The system has infinitely many solutions
- ☐ The system has exactly one solution

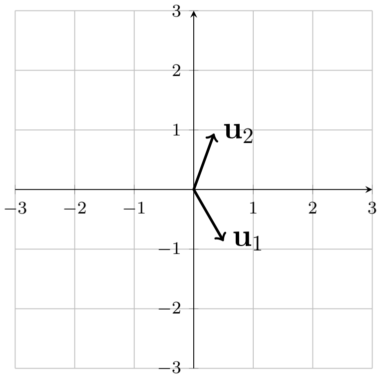
Problem 2 (5 points)

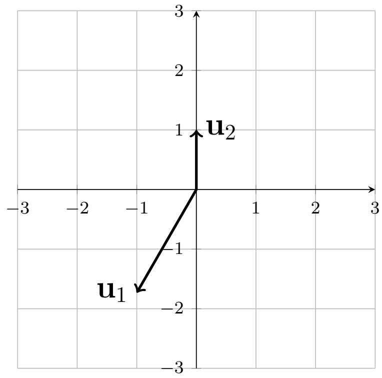
The figures below show two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^2$.

(a) Mark the figure where $\{\mathbf{u}_1, \mathbf{u}_2\}$ can be the result of applying Gram-Schmidt (with normalization) to a basis of $\mathbb{R}^2$.

☐ 

☐ 

☐ 

☐ 

Problem 3 (10 point)

The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 7 & -5 & 5 \\ -6 & 0 & 6 \\ 4 & -2 & 8 \end{bmatrix}$$

is $-(t-12)(t-6)(t+3)$.

(a) (2 points). Among the following numbers, the eigenvalue(s) of A is/are?

☐ -9 ☐ -6 ☐ -3 ☐ 0 ☐ 3 ☐ 6

(b) (2 points). Among the following vectors, the eigenvector(s) of A is/are?

☐ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) (1 point). Is A invertible?

☐ Yes ☐ No ☐ Neither yes nor no

(d) (1 point). Is A diagonalizable?

☐ Yes ☐ No ☐ Neither yes nor no

(e) (2 points). How many, not necessarily linearly independent, eigenvectors does A have?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ Infinitely many

(f) (2 points). If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, then EA:

☐ $\begin{bmatrix} 7 & -5 & 5 \\ -10 & 2 & -2 \\ 4 & -2 & 8 \end{bmatrix}$ ☐ $\begin{bmatrix} 7 & -5 & 5 \\ -6 & 0 & 6 \\ -2 & -2 & 14 \end{bmatrix}$

☐ $\begin{bmatrix} 7 & -5 & 5 \\ -6 & 0 & 6 \\ 10 & -2 & 2 \end{bmatrix}$ ☐ $\begin{bmatrix} 11 & -7 & 13 \\ -6 & 0 & 6 \\ 4 & -2 & 8 \end{bmatrix}$

Problem 4 (12 points)

Let $T\colon \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 5 & 7 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) (1 point). What is n?

☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

(b) (1 point). What is m?

☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

(c) (2 points). What is the rank $\text{Rank}(A)$?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

(d) (2 points). What is the nullity $\text{Nullity}(A)$?

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4

(e) (2 points). Is T injective (óne-to-óne) and/or surjective (onto)?

☐ Injective, but not surjective ☐ Injective and surjective
☐ Surjective, but not injective ☐ Neither injective nor surjective

(f) (2 points). The vector $a = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ is in:

☐ The column space $\text{Col}(A)$

☐ The null space $\text{Null}(A)$

☐ None of the above

(g) (2 points). The vector $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in:

☐ The column space $\text{Col}(A)$

☐ The null space $\text{Null}(A)$

☐ None of the above

Problem 5 (4 points)

A is an $n \times 3$-matrix, B is an $m \times 5$-matrix, and $C = AB$ is a $p \times p$-matrix.

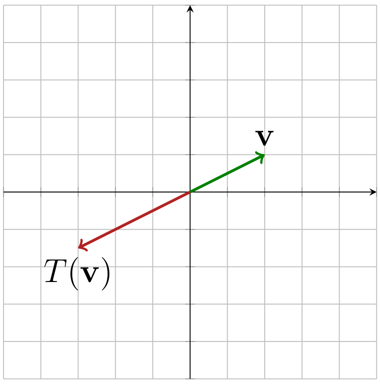
(a) What are the values of m, n and p?

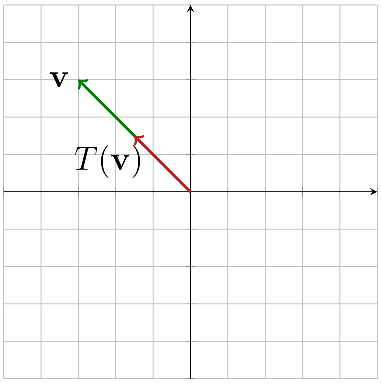
- ☐ $m = n = p = 3$
- ☐ $m = n = 5$, $p = 3$
- ☐ $m = 3$, $n = 5$, $p = 4$
- ☐ $m = n = p = 5$
- ☐ $m = 5$, $n = 3$, $p = 4$
- ☐ None of the previous

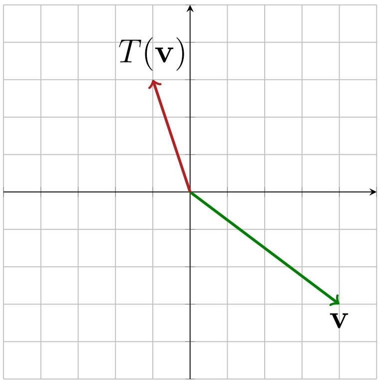
Problem 6 (5 points)

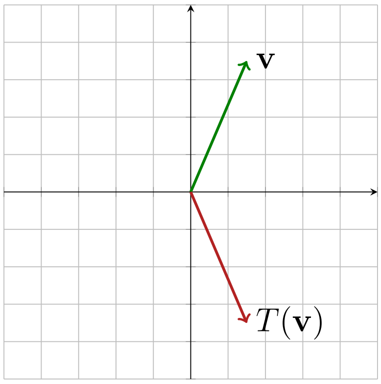
The figures below each show a vector $\mathbf{v}$ and its image under a linear map T. Note that the map T is *not* the same in every figure.

(a) Among the figures, the one(s) where $\mathbf{v}$ is an eigenvector of T is/are?

☐ 

☐ 

☐ 

☐ 

Problem 7 (10 point)

Let $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$.

(a) (5 points). What is $A\mathbf{v}$?

☐ $\begin{bmatrix} 16 \\ 2 \\ 5 \end{bmatrix}$ ☐ $\begin{bmatrix} -16 \\ 2 \\ -5 \end{bmatrix}$ ☐ $\begin{bmatrix} -16 \\ -18 \\ 5 \end{bmatrix}$ ☐ None of the previous

(b) (5 points). What is the inverse of A?

☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

Problem 8 (6 points)

Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{u} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$.

(a) (3 points). Are $\mathbf{v}_1$ and $\mathbf{v}_2$ orthogonal?

☐ Yes ☐ No ☐ Neither yes nor no

(b) (3 points). What is the orthogonal projection of $\mathbf{u}$ onto W?

☐ $\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix}$

☐ $\begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} -7 \\ 7 \\ 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 4 \\ 4 \\ 16 \end{bmatrix}$ ☐ $\begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix}$

Problem 9 (2 points)

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$. Mark those vectors below that are in Span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

☐ $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

Problem 10 (10 point)

Two matrices are given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By matrix multiplication, the matrix $C = AB$ is obtained.

(a) (3 points). What is the size of C?

☐ 2×3 ☐ 3×3 ☐ 2×4

☐ 3×2 ☐ 4×3 ☐ 4×2

(b) (3 points). What is entry c_{31}?

☐ -3 ☐ 0 ☐ 4

☐ -1 ☐ 1 ☐ 3

(c) (4 points). Which of the following products exist?

☐ AB ☐ A^TB ☐ B^TA ☐ A^TB^T

☐ BA ☐ BA^T ☐ AB^T ☐ B^TA^T

Problem 11 (10 point)

Let A and B be 3×3-matrices with determinants $\det(A) = 2$ and $\det(B) = 0$, respectively.

(a) (2 points). What is $\det(-A)$?

☐ 2 ☐ -2 ☐ -4 ☐ 4 ☐ 0 ☐ Not defined

(b) (2 points). What is $\det(AB^{-1})$?

☐ 2 ☐ -2 ☐ -4 ☐ 4 ☐ 0 ☐ Not defined

(c) (2 points). What is $\det(-B^2)$?

☐ 2 ☐ -2 ☐ -4 ☐ 4 ☐ 0 ☐ Not defined

(d) (2 points). What is $\det(B^T)$?

☐ 2 ☐ -2 ☐ -4 ☐ 4 ☐ 0 ☐ Not defined

(e) (2 points). What is $\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)$?

☐ 2 ☐ -2 ☐ -4 ☐ 4 ☐ 0 ☐ Not defined

Problem 12 (4 points)

A system of equations is given by

$$\begin{aligned} x_1 + 2x_2 + cx_3 &= 4 \\ x_1 + x_2 - 2x_3 - x_4 &= -2 \end{aligned}$$

where c is a real constant. Mark the true statement(s) below.

☐ $x_1 = -8$, $x_2 = 6$, $x_3 = 0$, $x_4 = 0$ is a solution regardless of the value of c.

☐ The system is consistent and has two free variables regardless of the value of c.

☐ The system is inconsistent regardless of the value of c.

☐ Whether the system is consistent or not depends on the value of c.

☐ If $c = 3$, then the system has exactly one solution, namely $x_1 = -8$, $x_2 = 6$, $x_3 = 0$, $x_4 = 0$.

Problem 13 (10 point)

The matrix A is row-reduced to the matrix R, where

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5\ \mathbf{a}_6]$ where $\mathbf{a}_i$ is the ith column in A.

(a) (2 points). Which of the following statements is correct?

- ☐ $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is linearly independent
- ☐ $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_6\}$ is linearly dependent.
- ☐ $\mathbf{a}_5 = \mathbf{a}_1 + 2\mathbf{a}_4$
- ☐ $\mathbf{a}_5 = -\mathbf{a}_1 - 2\mathbf{a}_4$
- ☐ $\mathbf{a}_6$ is a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

(b) (2 points). What is the nullity Nullity(A)?

- ☐ 0
- ☐ 1
- ☐ 2
- ☐ 3
- ☐ 4
- ☐ 5
- ☐ 6
- ☐ That cannot be determined with the given information.

(c) (2 points). What is the rank Rank(A)?

- ☐ 0
- ☐ 1
- ☐ 2
- ☐ 3
- ☐ 4
- ☐ 5
- ☐ 6
- ☐ That cannot be determined with the given information.

(d) (2 points). If A is the augmented matrix for a system of linear equations in $x_1, x_2, \ldots, x_5$, does the system have a solution?

- ☐ Yes
- ☐ No

(e) (2 points). Which columns in A are pivot columns?

- ☐ 2 and 5
- ☐ 5 and 6
- ☐ 2, 3, 4, and 5
- ☐ 1, 3, 4, and 6
- ☐ No columns
- ☐ All columns

Problem 14 (6 points)

In MATLAB's Command Window, the following is given as input:

```
>> u = [1; 0; 1; 0];
>> v = [1; 2; 2; 1];
>> w = [1; 2; 3; 4];
>> z = [1; 3; 2; 6];
>> T = [u v w z];
>> rref(T)
 ans =
         1    0    0    0
         0    1    0    0
         0    0    1    0
         0    0    0    1
```

If T is the augmented matrix for a system of linear equations $A\mathbf{x} = \mathbf{b}$, which of the following statements is then correct?

☐ The system has a unique solution: $\mathbf{x} = \mathbf{0}$.

☐ The system has no solutions

☐ The system may have no solutions, or it may have infinitely many; it depends on the value of $\mathbf{b}$