Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

14 January 2020, 9:00–13:00

This test consists of 9 numbered pages with 13 problems. For each problem there is given a number of points. The total for all problems is 100 points.

Allowed aids: Books, notes, xerox copies and print.

Not allowed: Electronic aids such as calculator or mathematical applications on the computer. Electronic documents.

Facit

Full points are given if all the correct and no wrong answers are checked. A wrong answer cancels a correct answer in the same question.

Problem 1 (10 points)

Two matrices are given by

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix multiplication results in the matrix $C = AB$.

(a) (5 points). What is the size of the matrix C?

- [x] 2×3
- [ ] 3×3
- [ ] 2×4
- [ ] 3×2
- [ ] 4×3
- [ ] 4×2

(b) (5 points). What is the entry c_{13}?

- [ ] -3
- [ ] 0
- [ ] 4
- [x] -1
- [ ] 1
- [ ] 3

Problem 2 (10 points)

Consider the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2 \\ 2x_1 + x_2 &= 1 \\ -x_1 + 2x_3 &= 1 \end{aligned}$$

Mark the two correct statements below:

- [ ] The system of equations has no solution
- [ ] $x_1 = x_2 = x_3 = 1$ is a solution to the system of equations
- [x] $x_1 = -1$, $x_2 = 3$, and $x_3 = 0$ is a solution to the system of equations
- [ ] The system of equations has exactly two solutions
- [x] The system of equations has infinitely many solutions
- [ ] $x_1 = -1$, $x_2 = 3$, and $x_3 = 0$ is the only solution to the system of equations

Problem 3 (10 points)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ og $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) (5 points). What is $A\mathbf{v}$?

☐ $\begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ☐ None of these

(b) (5 points). What is the inverse of A?

☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ ☐ None of these

Problem 4 (10 points)

The characteristic polynomial of the matrix

$$A = \begin{bmatrix} -6 & -1 & 8 \\ -4 & -3 & 8 \\ -4 & -6 & 11 \end{bmatrix}$$

is $-(t-3)(t-1)(t+2)$.

(a) (2 points). Which three of the following are eigenvalues of A?

☐ -3 ☑ -2 ☐ -1 ☐ 0 ☑ 1 ☐ 2 ☑ 3

(b) (2 points). Which one of the following is an eigenvector of A?

☐ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) (2 points). Is A invertible?

☑ Yes ☐ No

(d) (2 points). Is A diagonalisable?

☑ Yes ☐ No

(e) (2 points). If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, Then EA equals:

☑ $\begin{bmatrix} -6 & -1 & 8 \\ 0 & 3 & -3 \\ -4 & -6 & 11 \end{bmatrix}$ ☐ $\begin{bmatrix} -6 & -1 & 8 \\ -4 & -3 & 8 \\ 0 & -3 & 3 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 & -3 & 3 \\ -4 & -3 & 8 \\ -4 & -6 & 11 \end{bmatrix}$ ☐ $\begin{bmatrix} -6 & -1 & 8 \\ -4 & -3 & -8 \\ -4 & -6 & 11 \end{bmatrix}$

Problem 5 (12 points)

Let $T\colon \mathcal{R}^n \to \mathcal{R}^m$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 & 2 \\ 0 & 2 & 1 & 3 & 4 \\ 2 & 2 & 0 & 4 & 4 \end{bmatrix}.$$

(a) (2 points). What is n?

☐ 2 ☐ 3 ☐ 4 ☑ 5 ☐ 6

(b) (2 points). What is m?

☐ 2 ☑ 3 ☐ 4 ☐ 5 ☐ 6

(c) (2 points). What is the rank Rank(A)?

☑ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

(d) (2 points). What is the nullity Nullity(A)?

☐ 2 ☑ 3 ☐ 4 ☐ 5 ☐ 6

(e) (2 points). Does $a = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ belong to:

☑ The column space Col(A) ☐ The null space Null(A) ☐ None of these

(f) (2 points). Does $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ belong to:

☐ The column space Col(A) ☐ the null space Null(A) ☑ None of these

Problem 6 (2 points)

A is an $n \times 3$ matrix, B is an $m \times 5$ matrix, and $C = AB$ is a $p \times p$ matrix. What are the values of m, n, and p?

- ☐ $m = n = p = 3$
- ☐ $m = 3, n = 5, p = 4$
- ☐ $m = n = 5, p = 3$
- ☑ None of these

Problem 7 (6 points)

Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, and $\mathbf{u} = \begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix}$.

(a) (3 points). Are $\mathbf{v}_1$ and $\mathbf{v}_2$ orthogonal?

- ☐ Yes
- ☑ No
- ☐ Neither yes nor no

(b) (3 points). What is the orthogonal projection of $\mathbf{u}$ onto W?

- ☐ $\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$
- ☑ $\begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix}$
- ☐ $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$
- ☐ $\begin{bmatrix} -7 \\ 7 \\ 4 \end{bmatrix}$

Problem 8 (8 points)

The linear transformation $T\colon \mathcal{R}^2 \to \mathcal{R}^2$ satisfies that for $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $T(\mathbf{u}) = -\mathbf{u}$ and $T(\mathbf{v}) = \mathbf{v}$.

(a) (2 points). Are $T(\mathbf{u})$ and $T(\mathbf{v})$ orthogonal?

- ☑ Yes
- ☐ No

(b) (2 points). Which of the vectors $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors?

- ☐ Only $\mathbf{u}$
- ☐ Only $\mathbf{v}$
- ☑ Both $\mathbf{u}$ and $\mathbf{v}$
- ☐ None of them

(c) (2 points). Is it true that $T(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{R}^2$?

- ☑ Yes
- ☐ No

(d) (2 points). T is a

☑ Reflection ☐ Rotation ☐ None of these

Problem 9 (2 points)

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$. If, by applying the Gram-Schmidt process on $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, one gets orthogonal vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, then $\mathbf{v}_3$ equals:

☐ $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ☑ $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

Problem 10 (10 points)

Let A og B be 4×4 matrices with determinants $\det(A) = -4$ and $\det(B) = 2$, resp.

(a) (2 points). What is $\det(A^T)$?

☐ 2 ☐ -2 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

(b) (2 points). What is $\det(-A)$?

☐ 2 ☐ -2 ☑ -4 ☐ 4 ☐ -8 ☐ Not defined

(c) (2 points). What is $\det(AB^{-1})$?

☐ 2 ☑ -2 ☐ -4 ☐ 4 ☐ -8 ☐ Not defined

(d) (2 points). What is $\det(-B^2)$?

☐ 2 ☐ -2 ☐ -4 ☑ 4 ☐ -8 ☐ Not defined

(e) (2 points). What is $\det\left(\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 5 \end{bmatrix}\right)$?

☐ 2 ☐ -2 ☐ -4 ☑ 4 ☐ -8 ☐ Not defined

Problem 11 (10 points)

The matrix A is row reduced to the matrix R, where

$$R = \begin{bmatrix} 1 & 2 & 0 & -3 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5\ \mathbf{a}_6]$ where $\mathbf{a}_i$ is i'te column in A.

(a) (2 points). What is the nullity Nullity(A)?

- ☐ 0
- ☐ 1
- ☐ 2
- ☑ 3
- ☐ 4
- ☐ 5
- ☐ 6
- ☐ That can't be determined with the given information.

(b) (2 points). What is the rank Rank(A)?

- ☐ 0
- ☐ 1
- ☐ 2
- ☑ 3
- ☐ 4
- ☐ 5
- ☐ 6
- ☐ That can't be determined with the given information.

(c) (2 points). Which columns in A are pivot columns?

- ☐ $\mathbf{a}_1$ and $\mathbf{a}_2$
- ☐ $\mathbf{a}_2, \mathbf{a}_4$ and $\mathbf{a}_6$
- ☑ $\mathbf{a}_1, \mathbf{a}_3$ and $\mathbf{a}_5$
- ☐ $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$
- ☐ All of them.
- ☐ None of them.

(d) (2 points). Which of the following statements is correct?

- ☐ $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ are linearly independent.
- ☐ $\mathbf{a}_4 = 3\mathbf{a}_1 - 3\mathbf{a}_3$
- ☑ $\mathbf{a}_4 = -3\mathbf{a}_1 + 3\mathbf{a}_3$
- ☐ $\mathbf{a}_4 = 2\mathbf{a}_2 + 3\mathbf{a}_3$
- ☐ $\mathbf{a}_1 = 2\mathbf{a}_2 - 3\mathbf{a}_3$
- ☐ $\mathbf{a}_5$ is a linear combination of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

(e) (2 points). If $A = [B \ \mathbf{b}]$ is the augmented of a system of linear equations in $x_1, x_2, \dots, x_5$, does the system of equations then have a solution where $x_2 = x_4$?

- [x] Yes
- [ ] No
- [ ] That can't be determined with the given information.

Problem 12 (8 points)

A and B are two quadratic matrices, $\mathbf{v}$ is an eigenvector of A with eigenvalue 4, $\mathbf{w} = A\mathbf{v}$ is an eigenvector of B with eigenvalue -8.

(a) (2 points). Is $\mathbf{v}$ an eigenvector of BA?

- [ ] Yes, with eigenvalue -2
- [ ] Yes, with eigenvalue 4
- [ ] Yes, with eigenvalue -8
- [ ] Yes, with eigenvalue 64
- [x] Yes, with eigenvalue -32
- [ ] No

(b) (2 points). Is $\mathbf{v}$ an eigenvektor of AB?

- [ ] Yes, with eigenvalue -2
- [ ] Yes, with eigenvalue 4
- [ ] Yes, with eigenvalue -8
- [ ] Yes, with eigenvalue 64
- [x] Yes, with eigenvalue -32
- [ ] No

(c) (2 points). Is $\mathbf{v}$ an eigenvektor of B?

- [ ] Yes, with eigenvalue -2
- [ ] Yes, with eigenvalue 4
- [x] Yes, with eigenvalue -8
- [ ] Yes, with eigenvalue 64
- [ ] Yes, with eigenvalue -32
- [ ] No

(d) (2 points). Is $\mathbf{v}$ an eigenvektor of B^2?

- [ ] Yes, with eigenvalue -2
- [ ] Yes, with eigenvalue 4
- [ ] Yes, with eigenvalue -8
- [x] Yes, with eigenvalue 64
- [ ] Yes, with eigenvalue -32
- [ ] No

Problem 13 (2 points)

I MATLAB's Command Window the following is typed:

```
>> u = [1; 1; 1; 1];
>> v = [1; 2; 3; 4];
>> w = [1; 3; 6; 10];
>> z = [1; 4; 10; 19];
>> A = [u v w z];
>> rref(A)
 ans =
         1    0    0    1
         0    1    0   -3
         0    0    1    3
         0    0    0    0
```

If A is used as a coefficient matrix in a system of linear equations $A\mathbf{x} = \mathbf{b}$, which of the following statements is true?

☐ The system of equations has a unique solution: $x_1 = 1$, $x_2 = -3$ og $x_3 = 3$.

☐ The system of equations has infinitely many solutions; x_4 is a free variable

☑ Either the system of equations has no solutions, or it has infinitely many solutions; it depends on what $\mathbf{b}$ is.