For at finde den danske version af prøven, begynd venligst i den modsatte ende!

Please ignore the Danish version on the back, if you follow the English version of the exam.

Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

15 June 2018, 9:00 – 13:00

This test consists of 9 pages and 14 problems. All problems are "multiple choice" problems. For each problem there is given a number of points.
The total for all 14 problems is 100 points.

It is allowed to use books, notes, xerox copies etc.
It is **not allowed** to use **any electronic devices**.

Your answers must be given by marking the relevant boxes on these sheets.
The evaluation is only based on these markings.

In problems 1, 2 and 3 the evaluation is done following this principle:
Each wrong mark will annul one correct mark.

Remember to fill in your full name together with your student number below.

NAME:______________________________________

STUDENT NUMBER:______________________________________

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Problem 1 (6 points, with annulment)

Which of the statements about the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ 3x_1 + 7x_2 + 3x_3 &= 10 \\ x_1 + 3x_2 - x_3 &= 8 \end{aligned}$$

are true?

- ☑ $x_1 = 3, x_2 = 1, x_3 = -2$ solves the system.
- ☐ $x_1 = 2, x_2 = 2, x_3 = 1$ solves the system.
- ☑ The system has infinitely many solutions.

Problem 2 (8 points, with annulment)

In this problem one investigates the matrix

$$A = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & -1 \end{bmatrix}.$$

Which of the following claims are true?

- ☑ $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector for A.
- ☑ $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for A.
- ☑ $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector for A.
- ☐ $\mathbf{u}$ and $\mathbf{w}$ have the same corresponding eigenvalue.
- ☐ $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.
- ☑ A is diagonalisable.
- ☑ A is similar to the diagonal matrix $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.
- ☑ A is invertible (regular).

Problem 3 (8 points, with annulment)

A is a $m \times 2$-matrix and B is a $n \times 4$ matrix, such that $C = AB$ is a 4×4 matrix. A is the standard matrix of a linear transformation $S : \mathcal{R}^2 \to \mathcal{R}^m$, B is the standard matrix of a linear transformation $T : \mathcal{R}^4 \to \mathcal{R}^n$, and C is the standard matrix of a linear transformation $U : \mathcal{R}^4 \to \mathcal{R}^4$.

1. What are the values of m and n?

   ☐ $m = n = 2$ ☐ $m = 2, n = 4$ ☐ $m = n = 4$ ☑ $m = 4, n = 2$

   Which of the following statements can *never* be true?

2. ☑ S is onto (surjective)
   ☐ S is one-to-one (injective)

3. ☐ T is onto (surjective)
   ☑ T is one-to-one (injective)

4. ☑ U is onto (surjective)
   ☑ U is one-to-one (injective)

Problem 4 (8 points)

Given is a matrix $A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -1 & 3 & 6 \\ -1 & 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ 6 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix}$.

1. Is $\mathbf{d}$ contained in the column space Col A?

   ☐ Yes ☑ No

2. Is $\mathbf{b}$ contained in the column space Col A?

   ☑ Yes ☐ No

3. Does $\mathbf{c}$ belong to the null space Null A?

   ☐ Yes ☑ No

4. Does $\mathbf{d}$ belong to the null space Null A?

   ☐ Yes ☑ No

Problem 5 (4 points)

Which of the following vectors coincides with the product $A\mathbf{b}$ of the matrix $A = \begin{bmatrix} 3 & -1 \\ 5 & -3 \\ -5 & 2 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$?

☐ $\begin{bmatrix} 3 \\ -7 \end{bmatrix}$ ☐ $\begin{bmatrix} 5 \\ 4 \\ -6 \end{bmatrix}$ ☑ $\begin{bmatrix} 5 \\ 11 \\ -9 \end{bmatrix}$ ☐ none of them

Problem 6 (6 points)

This problem concerns the system

$$\begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 3 \\ 3x_1 & + & 2x_2 & - & 5x_3 & = & 11 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 7 \end{array}.$$

1. Which of the below matrices corresponds to the augmented matrix $[A \ \mathbf{b}]$?

   ☐ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 11 & -5 & -2 \\ 3 & 1 & 7 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & -5 & -2 \\ 3 & 11 & 7 \end{bmatrix}$

   ☐ $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -5 \\ 2 & 1 & -2 \end{bmatrix}$ ☑ $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 3 & 2 & -5 & 11 \\ 2 & 1 & -2 & 7 \end{bmatrix}$

2. Which of the below matrices equals the reduced echelon matrix that is row equivalent to $[A \ \mathbf{b}]$?

   ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   ☑ $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

3. Which of the following claims are true?

   ☐ The system is inconsistent.

   ☑ $x_1 = 2, x_2 = 5, x_3 = 1$ is one among several of the solutions of the system.

   ☐ $x_1 = 2, x_2 = 5, x_3 = 1$ is the only solution of the system.

   ☐ $x_1 = 1, x_2 = 3, x_3 = 3$ is one among several of the solutions of the system.

   ☐ $x_1 = 1, x_2 = 3, x_3 = 3$ is the only solution of the system.

Problem 7 (10 points)

In the equation $A\mathbf{x} = \mathbf{b}$ one has $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -2 & 4 & 6 & 4 \\ -1 & 2 & -1 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}$.

It is given for free that the augmented matrix $[A \ \ \mathbf{b}]$ is row equivalent to the reduced echelon matrix

$$\begin{bmatrix} 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Which columns in A are pivot columns?

   ☐ all four columns ☐ columns 1 and 4
   ☐ columns 2 and 4 ☑ columns 1 and 3

2. What is the rank of the matrix A?

   ☐ 0 ☐ 1 ☑ 2 ☐ 3 ☐ 4 ☐ 5

3. What is the nullity of the matrix A?

   ☐ 0 ☐ 1 ☑ 2 ☐ 3 ☐ 4 ☐ 5

4. Does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in which $x_2 = -3$?

   ☑ Yes ☐ No

5. The equation $A\mathbf{x} = \mathbf{b}$ has the particular solution

   ☐ $\mathbf{x} = (-2, 1, 1, 1)$ ☐ $\mathbf{x} = (-2, 1, -1, -1)$ ☑ $\mathbf{x} = (-2, 0, 1, 0)$

Problem 8 (6 points)

In this problem one investigates $\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -2 \\ 4 \\ t \end{bmatrix}$ in $\mathcal{R}^3$.

Notice that the last entry t in the vector $\mathbf{c}$ is an arbitrary real number.
Which of the following claims are correct?

1. $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linearly dependent vectors for

   ☐ $t = 2$ ☑ $t = -6$ ☐ $t \neq -6$ ☐ all real t

2. $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ span $\mathcal{R}^3$ for

   ☑ $t = 2$ ☐ $t = -6$ ☑ $t \neq -6$ ☐ all real t

Problem 9 (6 points)

Here is given two 5×5 matrices A and B. In addition $\mathbf{v}$ is an eigenvector for A with eigenvalue -3, and $\mathbf{w} = A\mathbf{v}$ is an eigenvector for B with eigenvalue 5.

1. Is $\mathbf{w}$ an eigenvector for $B^2 = BB$?

   ☑ Yes ☐ No

   In the affirmative case, what is the corresponding eigenvalue?

   ☐ -50 ☐ -25 ☑ 25 ☐ 50

2. Is $\mathbf{v}$ always an eigenvector for BA?

   ☑ Yes ☐ No

   In the affirmative case, what is the corresponding eigenvalue?

   ☐ -3 ☐ 5 ☑ -15 ☐ 15

3. Is $\mathbf{v}$ always an eigenvector for AB?

   ☑ Yes ☐ No

   In the affirmative case, what is the corresponding eigenvalue?

   ☐ -3 ☐ 5 ☑ -15 ☐ 15

Problem 10 (6 points)

The three vectors $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ og $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$ form a basis $\mathcal{B}$ for $\mathcal{R}^3$. By applying the Gram-Schmidt process to $\mathcal{B}$, one obtains an orthogonal basis consisting of the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. Here one has that:

1. ☑ $\mathbf{v}_1 = \mathbf{u}_1$ ☐ $\mathbf{v}_1 = -\mathbf{u}_1$ ☐ none of these
2. ☑ $\mathbf{v}_2 = \mathbf{u}_2 - \mathbf{u}_1$ ☐ $\mathbf{v}_2 = \mathbf{u}_2 + 2\mathbf{u}_1$ ☐ none of these
3. ☐ $\mathbf{v}_3 = \mathbf{u}_3$ ☐ $\mathbf{v}_3 = \mathbf{u}_3 + \mathbf{u}_1 + \mathbf{u}_2$
   ☐ $\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{u}_2$ ☑ none of these

Problem 11 (12 points)

In this problem one considers the following four matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Is it true that

1. A and B are inverses to each other?

   ☑ Yes ☐ No

2. C and D are inverses to each other?

   ☑ Yes ☐ No

3. A is diagonalisable?

   ☑ Yes ☐ No

4. B is diagonalisable?

   ☑ Yes ☐ No

5. C er diagonaliserbar?

   ☐ Yes ☑ No

6. D is diagonalisable?

   ☐ Yes ☑ No

Problem 12 (8 points)

This problem concerns the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ in $\mathcal{R}^2$ as well as a linear transformation $T : \mathcal{R}^2 \to \mathcal{R}^2$, which satisfies:

$$T(\mathbf{u}) = \mathbf{u} \text{ and } T(\mathbf{v}) = -\mathbf{v}.$$

Let $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ be the standard matrix of the transformation T (thus $\mathbf{a}_1$ and $\mathbf{a}_2$ are the column vectors in A).

1. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   ☑ Yes ☐ No

2. Which of the vectors $\mathbf{u}$, $\mathbf{v}$ are eigenvectors for T ?

   ☐ Only $\mathbf{u}$ ☐ Only $\mathbf{v}$ ☑ both $\mathbf{u}$ and $\mathbf{v}$ ☐ none of them

3. The vectors $\frac{1}{\sqrt{10}}\mathbf{u}$, $\frac{1}{\sqrt{10}}\mathbf{v}$ form an orthonomal basis for $\mathcal{R}^2$?

   ☑ Yes ☐ No

4. $\mathbf{a}_1 = \begin{bmatrix} -0,8 \\ 0,6 \end{bmatrix}$.

   ☑ Yes ☐ No

5. $\mathbf{a}_2 = \begin{bmatrix} 0,6 \\ 0,8 \end{bmatrix}$.

   ☑ Yes ☐ No

6. A is an orthogonal matrix.

   ☑ Yes ☐ No

7. Is $T(\mathbf{u}) \cdot T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathcal{R}^2$?

   ☑ Yes ☐ No

8. The transformation T describes in the plane

   ☑ a reflection ☐ a rotation ☐ none of these

Problem 13 (8 points)

Here is given 5×5-matrices A and B fulfilling $\det A = -3$ and $\det(AB) = 6$.

1. What is the value of $\det(-A)$?

   ☐ -3 ☑ 3 ☐ 15 ☐ 18

2. What is the value of $\det(2A)$?

   ☐ 6 ☐ 32 ☐ 96 ☑ -96

3. What is the value of $\det B$?

   ☐ 8 ☐ -8 ☐ 3 ☑ -2

4. What is the value of $\det((B^T A)^T)$?

   ☐ -6 ☑ 6 ☐ $\frac{3}{2}$ ☐ $-\frac{3}{2}$

Problem 14 (4 points)

The following commands are typed into MATLAB's Command Window:

```
>> A = [3 1 1; 1 2 -1; 2 -1 1 ];
>> b = [4; 3; 2];
>> T = [A b];
```

1. What is the size of the matrix T?

   ☐ 6×3 ☐ 3×3 ☐ 4×3 ☑ 3×4

2. The system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$. Which of the following combinations of MATLAB commands will calculate $\mathbf{x}$?

   ☐ `>> R = rref(T); x=R(4,:)`

   ☐ `>> R = rref(A); x=R(:,4)`

   ☑ `>> R = rref(T); x=R(:,4)`

   ☐ `>> R = rref(T); x=R(:,5)`

   ☐ `>> R = rref(A); x=R(4,:)`