Exam Linear Algebra: 14 January 2019

Full solutions

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2019

Problem 1 & 2

NB: All page numbers refer to the book compiled by O. Geil, "Elementary Linear Algebra".

Problem 1)

a) By the Definition of the matrix product on page 97, the product AB of a (3×4) matrix A with a (4×2) matrix B gives a (3×2) matrix.

b) By the same definition it follows that the c_{12} entry stems from the dot product of the second column of B with the first row vector of A, that is: $c_{12} = (3 \cdot 2) + (0 \cdot 3) + (1 \cdot -1) + (-1 \cdot 0) = 5$

Problem 2)

a) Apply cofactor expansion (theorem 3.1 on page 203) over for instance the last column of the matrix. Then its determinant is found as:

$$\begin{aligned}
& c \cdot \det \begin{bmatrix} 3 & 1 \\ -8 & -1 \end{bmatrix} - 0 + 3 \cdot \det \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \\
=& c((3 \cdot -1) - (-8 \cdot 1)) + 3((1 \cdot 1) - (3 \cdot 2)) \\
=& 5c - 15 = -10
\end{aligned}$$

b) A square matrix is not invertible when its determinant is equal to 0 (Theorem 3.4 on page 214). From the result above we see that the determinant equals $5c - 15$, which is equal to 0 for $c = 3$.

Problem 3

See the rules of theorem 3.3 on page 212,

THEOREM 3.3

Let A be an $n \times n$ matrix.

(a) If B is a matrix obtained by interchanging two rows of A, then $\det B = -\det A$.

(b) If B is a matrix obtained by multiplying each entry of some row of A by a scalar k, then $\det B = k \cdot \det A$.

(c) If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.

(d) For any $n \times n$ elementary matrix E, we have $\det EA = (\det E)(\det A)$.

thereby,

a) $\det(C) = 3 \cdot \det(A) = 6$

b) $\det(B^4) = (\det\ B)^4 = (-3)^4 = 81$

c) $\det(AB^{-1}A^{-1}) = \det(A)\det(B^{-1})\det(A^{-1}) = (\det\ A)(\frac{1}{\det\ B})(\frac{1}{\det\ A}) = (\frac{1}{\det\ B}) = -\frac{1}{3}$

Problem 4

Put the system in augmented matrix form, and bring this in reduced echelon form (see "Procedure for Solving a System of Linear Equations" the blue box on page 37):

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 3 & 1 & -1 & 8 \\ 2 & 2 & 5 & -13 \end{array}\right] \xrightarrow[r_3 = r_3 - 2r_1]{r_2 = r_2 - 3r_1} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & -11 & 2 & 5 \\ 0 & -6 & 7 & -15 \end{array}\right] \xrightarrow{r_2 = 6r_2 - 11r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 0 & -65 & 195 \\ 0 & -6 & 7 & -15 \end{array}\right] \xrightarrow{r_2 = -r_2/65} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & -6 & 7 & -15 \end{array}\right] \xrightarrow{r_3 = r_3 - 7r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & -6 & 0 & 6 \end{array}\right] \xrightarrow{r_3 = -r_3/6} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & -1 \end{array}\right] \xrightarrow[r_3 = r_2]{r_2 = r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{r_1 = r_1 - 4r_2 + r3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Unique solution $x_1 = 2, x_2 = -1, x_3 = -3$.

Problem 5

Put the system in augmented matrix form, and bring it in reduced echelon form (see "Procedure for Solving a System of Linear Equations" the blue box on page 37):

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 3 & r & 2 \\ 3 & 2 & 1 & 8 \end{array}\right] \xrightarrow[r_3=r_3-3r_1]{r_2=r_2-r_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 4 & r-2 & 1 \\ 0 & 5 & -5 & 5 \end{array}\right] \xrightarrow[r_3=r_2]{r_2=r_3/5}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & r-2 & 1 \end{array}\right] \xrightarrow[r_3=r_3-4r_2]{r_1=r_1+r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & r+2 & -3 \end{array}\right]$$

From the final matrix we observe that the system is inconsistent for $r = -2$ as in that case we have a zero row on the left of the vertical augmentation line while a nonzero entry on the right.

Note: For $r = -2$ the final matrix is in reduced echelon form; while for $r \neq -2$ we should divide the third row by $r + 2$, and perform $r_2 = r_2 + r_3$ and $r_1 = r_1 - r_3$ to get the matrix in reduced echelon form.

Problem 6

a) The standard matrix of S is (theorem 2.9 on page 174):
$[S(\mathbf{e}_1)\ S(\mathbf{e}_2)] = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$

b) Composite transformation $ST : R^2 \to R^2$ states first apply T and then S. By theorem 2.12 on page 186 the standard matrix of ST is found as the matrix product $A_S A_T$ with A_T the standard matrix of T and A_S the standard matrix of S.
The standard matrix of T is $[T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$ and thereby

$$A_S A_T = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 17 & 14 \end{bmatrix}$$

c) The standard matrix of the inverse transform of T equals the inverse of standard matrix of T (see theorem 2.13 on page 187). For an invertible 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ its inverse is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (see the boxed result on page 200) so the inverse of the standard matrix $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$ of T is: $\begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$

Problem 7

The eigenvalues of a square matrix A are the values of λ that satisfy $\det(A - \lambda I_n) = 0$, see the boxed result on page 302.

Hence we find the eigenvalues of the matrix by that:

$$\begin{aligned}\det\left(\begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix} - \lambda I_2\right) &= \det\left(\begin{bmatrix} 1-\lambda & 4 \\ -1 & 5-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(5-\lambda) - (-1 \cdot 4) \\ &= \lambda^2 - 6\lambda + 9 \\ &= (\lambda - 3)^2,\end{aligned}$$

which is zero for $\lambda = 3$ with multiplicity 2 (for the definition of multiplicity see page 305).

Problem 8

a) A straightforward strategy is to calculate $A\mathbf{v_1}, A\mathbf{v_2}, A\mathbf{v_3}, A\mathbf{v_4}$ and $A\mathbf{v_5}$ and to notice that $A\mathbf{v_1} = 3\mathbf{v}_1$, $A\mathbf{v_4} = 3\mathbf{v}_4$, $A\mathbf{v_5} = 3\mathbf{v_5}$, while $A\mathbf{v_2}$ and $A\mathbf{v_3}$ are not fixed multiples of respectively, $\mathbf{v}_2$ and $\mathbf{v}_3$. So $\mathbf{v}_1, \mathbf{v}_4$ and $\mathbf{v}_5$ are eigenvectors while $\mathbf{v}_2$ and $\mathbf{v_3}$ are not.

b) From the characteristic polynomial (see page 302 for the definition and its significance) of A it is immediately clear that is only real eigenvalue is $\lambda = 3$.

c) From the characteristic polynomial we notice that there are complex eigenvalues, therefore the matrix is not diagonalizable (see the blue box on page 319, "Test for a Diagonalizable Matrix ...")

Problem 9

Start from the definition of the matrix representation of T with respect to β om page 277, then,

$$[T]_\beta = \Big[[T(\mathbf{b}_1)]_\beta \; [T(\mathbf{b}_2)]_\beta \; [T(\mathbf{b}_3)]_\beta \Big]$$
$$= \Big[B^{-1} T(\mathbf{b}_1) \; B^{-1} T(\mathbf{b}_2) \; B^{-1} T(\mathbf{b}_3) \Big]$$
$$= B^{-1} \Big[T(\mathbf{b}_1) \; T(\mathbf{b}_2) \; T(\mathbf{b}_3) \Big]$$

with

$$B = [\mathbf{b}_1 \; \mathbf{b}_2 \; \mathbf{b}_3] = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -2 \\ -2 & -2 & -3 \end{bmatrix}$$

while we obtain $T(\mathbf{b}_1)$, $T(\mathbf{b}_2)$, $T(\mathbf{b}_3)$ from the transformation rule

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_3 \\ 2x_1 \end{bmatrix} \implies T(\mathbf{b}_1) = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} ; T(\mathbf{b}_2) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} ; T(\mathbf{b}_3) = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

Now we can compute $B^{-1}\Big[T(\mathbf{b}_1) \; T(\mathbf{b}_2) \; T(\mathbf{b}_3) \Big]$ for instance by applying the algorithm for computing $A^{-1}B$ stated on page 139, that is ...

Problem 9

... we augment B with the matrix $[T(\mathbf{b}_1)\ T(\mathbf{b}_2)\ T(\mathbf{b}_3)]$ to get

$$[\,B \mid T(\mathbf{b}_1)\ T(\mathbf{b}_2)\ T(\mathbf{b}_3)]$$

and bring it by Gausssian elimination steps into the form

$$\left[I_3 | B^{-1}\,[T(\mathbf{b}_1)\ T(\mathbf{b}_2)\ T(\mathbf{b}_3)]\right].$$

as shown below:

$$[\,B \mid T(\mathbf{b}_1)\ T(\mathbf{b}_2)\ T(\mathbf{b}_3)] = \left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & -2 & 3 & 3 & 5 \\ -2 & -2 & -3 & 2 & 2 & 4 \end{array}\right] \xrightarrow[r_3=r_3+2r_2]{r_2=r_2+r_1}$$

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 4 & 4 \end{array}\right] \xrightarrow{r_1=r_1-r_2+2r_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -7 & -11 & -13 \\ 0 & 1 & 0 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 4 & 4 \end{array}\right] = \left[I_3 \mid [T]_\beta\right]$$

Problem 10

By the Gram–Scmidt procedure (Theorem 6.6 on page 378) an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is found as

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 = [1\ 0\ -1\ 0]^T \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
&= \mathbf{u}_2 - \frac{(1\cdot 1) + (1\cdot 0) + (-3\cdot -1) + (1\cdot 0)}{(1\cdot 1) + (0\cdot 0) + (-1\cdot -1) + (0\cdot 0)}\mathbf{v}_1 \\
&= \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \\
\mathbf{v}_3 &= \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&= \mathbf{u}_3 - \frac{-8}{2}\mathbf{v}_1 - \frac{-8}{4}\mathbf{v}_2 = \mathbf{u}_3 + 4\mathbf{v}_1 + 2\mathbf{v}_2 \\
&= \begin{bmatrix} 0 \\ -1 \\ 8 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}$$

Problem 11

a) Notice that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for W as $||\mathbf{v}_1|| = ||\mathbf{v}_2|| = 1$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Then by Theorem 6.7 on page 392 we find the orthogonal projection $\mathbf{w}$ of $\mathbf{u}$ on W as

$$\begin{aligned} \mathbf{w} &= (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{u} \cdot \mathbf{v}_2)\mathbf{v}_2 \\ &= 3\mathbf{v}_1 + 3\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} \end{aligned}$$

b) The distance from $\mathbf{u}$ to W is $||\mathbf{z}||$ (see Theorem 6.7 as well as the blue box on page 397)

$$\mathbf{z} = \mathbf{u} - \mathbf{w} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -2 \end{bmatrix}; ||\mathbf{z}|| = \sqrt{2^2 + (-1)^2 + 0^2 + (-2)^2} = 3$$

c) For any subspace W of R^n: $\dim W + \dim W^\perp = n$ (see boxed result on page 393). So here: $\dim W^\perp = 4 - 2 = 2$

Problem 12

It is stated that $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are in R^3 and that:

(1) $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent,

(2) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not

which implies that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, which equivalently can be stated as that $\mathbf{v}_3$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$. So a basis for W is given by $\{\mathbf{v}_1, \mathbf{v}_2\}$ and thereby dim $W = 2$ and the subspace of W can be described as a 2- dimensional plane through the origin of R^3.

Observe that A is a square 3×3 matrix. Now (1) and (2) immediately imply that rank A is 2 (as rank A is the number of linear independent colums in A), which is not equal to the number of columns (i.e. A has not a full rank) and therefore A is not invertible and equivalently has a determinant equal to zero (see theorem 2.6 on page 138).

Hence from the given statements the only statement that is correct, is that $\det(A) = 0$.

Problem 13

a)
- $\{\mathbf{v}_1, \mathbf{v}_2\}$ is not linear independent as $\mathbf{v}_2 = 3\mathbf{v}_1$
- $\{\mathbf{v}_1, \mathbf{v}_3\}$ **is linear independent** as $\mathbf{v}_3$ is not a fixed multiple of $\mathbf{v}_1$
- $\{\mathbf{v}_1\}$ **is linear independent** (see the definition of linear independence on page 75/76)
- $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ can not be linear independent as each $\mathbf{v}_i \in R^3$, so only a maximum number of 3 of those vectors can be linear independent (see theorem 4.6 page 246)
- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ can not be linear independent for the same reason as stated immediately above
- to check if $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ is linear independent we put these vectors as column vectors in a matrix and find its reduced echelon form. This turns out to be the the identity matrix, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ **is linear independent**

b) To span R^3, the set needs to contain 3 or more vectors of which 3 are linear independent (see e.g. the boxed result on page 244). $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ do not contain 3 linear independent vectors as $\mathbf{v}_2 = 3\mathbf{v}_1$, while $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ **does** contain 3 linear independent vectors as we have verified in (a). So of the given sets the only set that spans R^3 is $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$

Problem 14

With $t = \pi/4$ and $A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ then

$$A = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Hence with $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} AA\mathbf{v} &= \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$