Exam in Linear Algebra

First Year at the Faculty of Engineering and Science
and the Technical Faculty of IT and Design

14 January 2019, 9:00-13:00

The present exam set consists of 8 numbered pages with 14 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100. The exam is held as a digital exam.

Allowed aid: Books, notes, photocopies and prints.

Not allowed: Electronic aid such as calculator or mathematics program on the computer. Electronic documents.

See also the general guidelines for the exam.

Good luck!

Problem 1 (5 points)

Two matrices are given as

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -2 & 3 & 7 \\ 1 & 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 2 & 1 \\ 5 & -1 \end{bmatrix}.$$

By matrix multiplication one gets the matrix $C = AB$.

(a) (2 points). What is the size of matrix C?

☐ 2×3 ☐ 3×4 ☐ 2×4

☐ 3×2 ☐ 4×3 ☐ 4×2

(b) (3 points). What is the entry c_{12}?

☐ -3 ☐ 0 ☐ 3

☐ -1 ☐ 1 ☐ 5

Problem 2 (6 points)

A matrix is defined as

$$\begin{bmatrix} 1 & 2 & c \\ 3 & 1 & 0 \\ -8 & -1 & 3 \end{bmatrix},$$

where c is a real constant.

(a) (3 points). What is the determinant of the matrix when $c = 1$?

☐ -15 ☐ -10 ☐ 3 ☐ 12

☐ -12 ☐ -8 ☐ 5 ☐ 14

(b) (3 points). For which value of c is the matrix not invertible?

☐ -8 ☐ -3 ☐ 0 ☐ 3

☐ -7 ☐ -11 ☐ 2 ☐ 5

Problem 3 (6 points)

Let A and B be two 3×3-matrices with determinants

$$\det(A) = 2, \quad \det(B) = -3.$$

Furthermore, let C be the matrix which appears by performing the elementary row operation $5r_1 + r_2 \to r_2$ followed by the operation $3r_2 \to r_2$ on A.

(a) (2 points). What is $\det(C)$?

☐ -18	☐ -6	☐ 3	☐ 10
☐ -12	☐ 2	☐ 6	☐ 12

(b) (2 points). What is $\det(B^4)$?

☐ -64	☐ -12	☐ 10	☐ 81
☐ -40	☐ -8	☐ 50	☐ 142

(c) (2 points). What is $\det(AB^{-1}A^{-1})$?

☐ -12	☐ -3	☐ $\frac{1}{2}$	☐ $\frac{10}{3}$
☐ -6	☐ $-\frac{1}{3}$	☐ 5	☐ 12

Problem 4 (6 points)

A system of linear equations is given by

$$\begin{aligned} x_1 + 4x_2 - x_3 &= 1 \\ 3x_1 + x_2 - x_3 &= 8 \\ 2x_1 + 2x_2 + 5x_3 &= -13. \end{aligned}$$

Mark the correct statement below.

☐ The only solution of the system is $x_1 = 2$, $x_2 = 1$, $x_3 = 5$.

☐ There are infinitely many solutions. One of these is $x_1 = 2$, $x_2 = 1$, $x_3 = 5$.

☐ The only solution of the system is $x_1 = 2$, $x_2 = -1$, $x_3 = -3$.

☐ There are infinitely many solutions. One of these is $x_1 = 2$, $x_2 = -1$, $x_3 = -3$.

☐ The system is inconsistent.

☐ The system is consistent and it has two free variables.

Problem 5 (8 points)

A system of equations is given by

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_1 + 3x_2 + rx_3 &= 2 \\ 3x_1 + 2x_2 + x_3 &= 8, \end{aligned}$$

where r is a real constant. For which value of r is the system inconsistent?

☐ -7	☐ -2	☐ 0	☐ 5
☐ -5	☐ $-\frac{1}{3}$	☐ 3	☐ 12

Problem 6 (9 points)

Let $S : \mathcal{R}^2 \to \mathcal{R}^2$ and $T : \mathcal{R}^2 \to \mathcal{R}^2$ be the linear transformations with transformation rules

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix}, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + 4x_2 \\ x_1 + x_2 \end{bmatrix}.$$

(a) (3 points). What is the standard matrix of S?

☐ $\begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix}$	☐ $\begin{bmatrix} -2 & 2 \\ 1 & 3 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
☐ $\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) (3 points). What is the standard matrix of the composite transformation $ST : \mathcal{R}^2 \to \mathcal{R}^2$?

☐ $\begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$	☐ $\begin{bmatrix} 3 & 2 \\ 17 & 14 \end{bmatrix}$
☐ $\begin{bmatrix} 5 & -8 \\ 3 & 2 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & 15 \\ -12 & 2 \end{bmatrix}$

(c) (3 points). What is the standard matrix of the inverse transformation $T^{-1} : \mathcal{R}^2 \to \mathcal{R}^2$?

☐ $\begin{bmatrix} 5 & -1 \\ 4 & -1 \end{bmatrix}$	☐ $\begin{bmatrix} 2 & 1 \\ -3 & 7 \end{bmatrix}$	☐ $\begin{bmatrix} 1 & -1 \\ 3 & 8 \end{bmatrix}$
☐ $\begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$	☐ $\begin{bmatrix} 2 & 10 \\ -8 & 4 \end{bmatrix}$	☐ $\begin{bmatrix} 7 & 4 \\ -21 & 2 \end{bmatrix}$

Problem 7 (6 points)

Consider the matrix

$$\begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}.$$

What are its eigenvalues?

- ☐ 2 and 4
- ☐ -1 and 1
- ☐ 3 and -5
- ☐ $\frac{1}{2}$ and 5
- ☐ 3 with multiplicity 2
- ☐ there are none

Remark: In the problem below there are one or more correct answers in question (a) and (b). Here one false mark will cancel one correct mark.

Problem 8 (9 points)

This problem is about matrices with real entries. Such a matrix is defined as

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}.$$

The characteristic polynomial of A is

$$-(t-3)(t^2+1).$$

Furthermore, a list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

(a) (3 points, with annulment). Mark those vectors below which are eigenvectors of A.

☐ $\mathbf{v}_1$ ☐ $\mathbf{v}_2$ ☐ $\mathbf{v}_3$ ☐ $\mathbf{v}_4$ ☐ $\mathbf{v}_5$

(b) (3 points, with annulment). Mark the eigenvalues of A.

☐ 0 ☐ 1 ☐ -3 ☐ 2 ☐ 3

(c) (3 points). Is A a diagonalizable matrix?

☐ Yes ☐ No

Problem 9 (9 points)

A basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $\mathcal{R}^3$ is given by

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}.$$

Furthermore, a linear operator T on $\mathcal{R}^3$ is defined by the transformation rule

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_3 \\ 2x_1 \end{bmatrix}.$$

What is the matrix representation $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$?

☐ $\begin{bmatrix} -1 & 1 & 3 \\ 2 & 14 & -7 \\ 9 & 12 & -3 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 2 & 4 & 4 \end{bmatrix}$

☐ $\begin{bmatrix} -7 & -11 & -13 \\ 3 & 4 & 5 \\ 2 & 4 & 4 \end{bmatrix}$ ☐ $\begin{bmatrix} 2 & -2 & 8 \\ 0 & 1 & 17 \\ 2 & -4 & 3 \end{bmatrix}$

☐ $\begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & -11 \\ 2 & 0 & 1 \end{bmatrix}$ ☐ $\begin{bmatrix} -11 & -8 & 10 \\ 1 & 0 & 5 \\ 9 & 12 & -3 \end{bmatrix}$

Problem 10 (8 points)

The subspace W of $\mathcal{R}^4$ has basis $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 8 \\ 1 \end{bmatrix}.$$

By applying the Gram-Schmidt process on $\mathcal{B}$ one finds an orthogonal basis for W. Mark this basis below.

☐ $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}\right\}$ ☐ $\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

☐ $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 3 \\ -3 \end{bmatrix}\right\}$ ☐ $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

Problem 11 (11 points)

Three vectors are given by

$$\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} 0 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

The subspace W of $\mathcal{R}^4$ is defined as $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) (4 points). What is the orthogonal projection of $\mathbf{u}$ on W?

□ $\begin{bmatrix} 2 \\ 4 \\ 5 \\ 0 \end{bmatrix}$ □ $\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$ □ $\begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ □ $\begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$

(b) (4 points). What is the distance from $\mathbf{u}$ to W?

□ 1 □ $3\sqrt{2}$ □ 5 □ $\sqrt{3}$

□ $\sqrt{2}$ □ $\sqrt{5}$ □ 3 □ $\sqrt{11}$

(c) (3 points). What is the dimension of $W^{\perp}$?

□ 0 □ 2 □ 4 □ 10

□ 1 □ 3 □ 8 □ 12

Problem 12 (5 points)

Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ be vectors in $\mathcal{R}^3$ such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Define a subspace and a matrix as

$$W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}, \quad A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3].$$

Mark the correct statement below.

□ The subspace W can be described as a line through the origin of $\mathcal{R}^3$.

□ The dimension of the subspace W is 3.

□ The linear combination $\mathbf{v}_1 - \mathbf{v}_2 + 5\mathbf{v}_3$ does not belong to W.

□ A is an invertible matrix.

□ $\det(A) = 0$.

Remark: In the problem below there are one or more correct answers in question (a) and (b). Here one false mark will cancel one correct mark.

Problem 13 (6 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (3 points, with annulment). Which of the following sets are linearly independent?

- ☐ $\{\mathbf{v}_1, \mathbf{v}_2\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_3\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$
- ☐ $\{\mathbf{v}_1\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$

(b) (3 points, with annulment). Which of the following sets span $\mathcal{R}^3$?

- ☐ $\{\mathbf{v}_1, \mathbf{v}_2\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_3\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$
- ☐ $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$
- ☐ $\{\mathbf{v}_2\}$

Problem 14 (6 points)

In MATLAB's Command Window one types the following:

```
>> t = pi/4;
>> A = [ cos(t) -sin(t) ; sin(t) cos(t) ];
>> v = [1; 1];
>> A*A*v
```

What answer will MATLAB give?

☐
```
ans =
        1.0000
        2.0000
```

☐
```
ans =
       -1.0000
        1.0000
```

☐
```
ans =
        1.0000
        1.0000
```

☐
```
ans =
        1.5000
       -1.5000
```

☐
```
ans =
        0.0000
        0.0000
```

☐
```
ans =
        2.0000
       -2.5000
```