

# Exam in Linear Algebra

First Year at the Faculty of Engineering and Science  
and the Technical Faculty of IT and Design

14 January 2019, 9:00-13:00

The present exam set consists of 8 numbered pages with 14 multiple-choice problems. A number of points are specified for each question. The total number of points equals 100. The exam is held as a digital exam.

**Allowed aid:** Books, notes, photocopies and prints.

**Not allowed:** Electronic aid such as calculator or mathematics program on the computer. Electronic documents.

See also the general guidelines for the exam.

Good luck!

## Answers

### Problem 1 (5 points)

Two matrices are given as

$$A = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -2 & 3 & 7 \\ 1 & 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 2 & 1 \\ 5 & -1 \end{bmatrix}.$$

By matrix multiplication one gets the matrix  $C = AB$ .

(a) (2 points). What is the size of matrix  $C$ ?

$2 \times 3$

$3 \times 4$

$2 \times 4$

$3 \times 2$

$4 \times 3$

$4 \times 2$

(b) (3 points). What is the entry  $c_{12}$ ?

$-3$

$0$

$3$

$-1$

$1$

$5$

### Problem 2 (6 points)

A matrix is defined as

$$\begin{bmatrix} 1 & 2 & c \\ 3 & 1 & 0 \\ -8 & -1 & 3 \end{bmatrix},$$

where  $c$  is a real constant.

(a) (3 points). What is the determinant of the matrix when  $c = 1$ ?

$-15$

$-10$

$3$

$12$

$-12$

$-8$

$5$

$14$

(b) (3 points). For which value of  $c$  is the matrix not invertible?

$-8$

$-3$

$0$

$3$

$-7$

$-11$

$2$

$5$

### Problem 3 (6 points)

Let  $A$  and  $B$  be two  $3 \times 3$ -matrices with determinants

$$\det(A) = 2, \quad \det(B) = -3.$$

Furthermore, let  $C$  be the matrix which appears by performing the elementary row operation  $5r_1 + r_2 \rightarrow r_2$  followed by the operation  $3r_2 \rightarrow r_2$  on  $A$ .

(a) (2 points). What is  $\det(C)$ ?

- |                              |                             |                                       |                             |
|------------------------------|-----------------------------|---------------------------------------|-----------------------------|
| <input type="checkbox"/> -18 | <input type="checkbox"/> -6 | <input type="checkbox"/> 3            | <input type="checkbox"/> 10 |
| <input type="checkbox"/> -12 | <input type="checkbox"/> 2  | <input checked="" type="checkbox"/> 6 | <input type="checkbox"/> 12 |

(b) (2 points). What is  $\det(B^4)$ ?

- |                              |                              |                             |  |
|------------------------------|------------------------------|-----------------------------|--|
| <input type="checkbox"/> -64 | <input type="checkbox"/> -12 | <input type="checkbox"/> 10 | <input checked="" type="checkbox"/> 81 |
| <input type="checkbox"/> -40 | <input type="checkbox"/> -8  | <input type="checkbox"/> 50 | <input type="checkbox"/> 142           |

(c) (2 points). What is  $\det(AB^{-1}A^{-1})$ ?

- |                              |  |  |   |
|------------------------------|--|--|---|
| <input type="checkbox"/> -12 | <input type="checkbox"/> -3                        | <input type="checkbox"/> $\frac{1}{2}$ | <input type="checkbox"/> $\frac{10}{3}$ |
| <input type="checkbox"/> -6  | <input checked="" type="checkbox"/> $-\frac{1}{3}$ | <input type="checkbox"/> 5             | <input type="checkbox"/> 12             |

### Problem 4 (6 points)

A system of linear equations is given by

$$\begin{aligned}x_1 + 4x_2 - x_3 &= 1 \\3x_1 + x_2 - x_3 &= 8 \\2x_1 + 2x_2 + 5x_3 &= -13.\end{aligned}$$

Mark the correct statement below.

- The only solution of the system is  $x_1 = 2, x_2 = 1, x_3 = 5$ .
- There are infinitely many solutions. One of these is  $x_1 = 2, x_2 = 1, x_3 = 5$ .
- The only solution of the system is  $x_1 = 2, x_2 = -1, x_3 = -3$ .
- There are infinitely many solutions. One of these is  $x_1 = 2, x_2 = -1, x_3 = -3$ .
- The system is inconsistent.
- The system is consistent and it has two free variables.

### Problem 5 (8 points)

A system of equations is given by

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 1 \\x_1 + 3x_2 + rx_3 &= 2 \\3x_1 + 2x_2 + x_3 &= 8,\end{aligned}$$

where  $r$  is a real constant. For which value of  $r$  is the system inconsistent?

- 7                       -2                       0                       5  
 -5                        $-\frac{1}{3}$                        3                       12

### Problem 6 (9 points)

Let  $S : \mathcal{R}^2 \rightarrow \mathcal{R}^2$  and  $T : \mathcal{R}^2 \rightarrow \mathcal{R}^2$  be the linear transformations with transformation rules

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \end{bmatrix}, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + 4x_2 \\ x_1 + x_2 \end{bmatrix}.$$

(a) (3 points). What is the standard matrix of  $S$ ?

- $\begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix}$                         $\begin{bmatrix} -2 & 2 \\ 1 & 3 \end{bmatrix}$                         $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$   
  $\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$                         $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$                         $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) (3 points). What is the standard matrix of the composite transformation  $ST : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ ?

- $\begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix}$                         $\begin{bmatrix} 1 & -1 \\ 2 & 5 \end{bmatrix}$                         $\begin{bmatrix} 3 & 2 \\ 17 & 14 \end{bmatrix}$   
  $\begin{bmatrix} 5 & -8 \\ 3 & 2 \end{bmatrix}$                         $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$                         $\begin{bmatrix} 1 & 15 \\ -12 & 2 \end{bmatrix}$

(c) (3 points). What is the standard matrix of the inverse transformation  $T^{-1} : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ ?

- $\begin{bmatrix} 5 & -1 \\ 4 & -1 \end{bmatrix}$                         $\begin{bmatrix} 2 & 1 \\ -3 & 7 \end{bmatrix}$                         $\begin{bmatrix} 1 & -1 \\ 3 & 8 \end{bmatrix}$   
  $\begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$                         $\begin{bmatrix} 2 & 10 \\ -8 & 4 \end{bmatrix}$                         $\begin{bmatrix} 7 & 4 \\ -21 & 2 \end{bmatrix}$

### Problem 7 (6 points)

Consider the matrix

$$\begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}.$$

What are its eigenvalues?

- 2 and 4                       -1 and 1  
 3 and -5                      $\frac{1}{2}$  and 5  
 3 with multiplicity 2        there are none

Remark: In the problem below there are one or more correct answers in question (a) and (b). Here one false mark will cancel one correct mark.

### Problem 8 (9 points)

This problem is about matrices with real entries. Such a matrix is defined as

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}.$$

The characteristic polynomial of  $A$  is

$$-(t - 3)(t^2 + 1).$$

Furthermore, a list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

(a) (3 points, with annulment). Mark those vectors below which are eigenvectors of  $A$ .

- $\mathbf{v}_1$                 $\mathbf{v}_2$                 $\mathbf{v}_3$                 $\mathbf{v}_4$                 $\mathbf{v}_5$

(b) (3 points, with annulment). Mark the eigenvalues of  $A$ .

- 0               1               -3               2               3

(c) (3 points). Is  $A$  a diagonalizable matrix?

- Yes                                       No

**Problem 9 (9 points)**

A basis  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$  for  $\mathcal{R}^3$  is given by

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}.$$

Furthermore, a linear operator  $T$  on  $\mathcal{R}^3$  is defined by the transformation rule

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_3 \\ 2x_1 \end{bmatrix}.$$

What is the matrix representation  $[T]_{\mathcal{B}}$  of  $T$  with respect to the basis  $\mathcal{B}$ ?

$\begin{bmatrix} -1 & 1 & 3 \\ 2 & 14 & -7 \\ 9 & 12 & -3 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 2 & 4 & 4 \end{bmatrix}$

$\begin{bmatrix} -7 & -11 & -13 \\ 3 & 4 & 5 \\ 2 & 4 & 4 \end{bmatrix}$

$\begin{bmatrix} 2 & -2 & 8 \\ 0 & 1 & 17 \\ 2 & -4 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 2 \\ 4 & 5 & -11 \\ 2 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -11 & -8 & 10 \\ 1 & 0 & 5 \\ 9 & 12 & -3 \end{bmatrix}$

**Problem 10 (8 points)**

The subspace  $W$  of  $\mathcal{R}^4$  has basis  $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ , where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 8 \\ 1 \end{bmatrix}.$$

By applying the Gram-Schmidt process on  $\mathcal{B}$  one finds an orthogonal basis for  $W$ . Mark this basis below.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 3 \\ -3 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

### Problem 11 (11 points)

Three vectors are given by

$$\mathbf{v}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

The subspace  $W$  of  $\mathcal{R}^4$  is defined as  $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ .

(a) (4 points). What is the orthogonal projection of  $\mathbf{u}$  on  $W$ ?

- $\begin{bmatrix} 2 \\ 4 \\ 5 \\ 0 \end{bmatrix}$         $\begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$         $\begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$         $\begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$

(b) (4 points). What is the distance from  $\mathbf{u}$  to  $W$ ?

- 1        $3\sqrt{2}$        5        $\sqrt{3}$   
  $\sqrt{2}$         $\sqrt{5}$        3        $\sqrt{11}$

(c) (3 points). What is the dimension of  $W^\perp$ ?

- 0       2       4       10  
 1       3       8       12

### Problem 12 (5 points)

Let  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$  be vectors in  $\mathcal{R}^3$  such that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly independent and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent. Define a subspace and a matrix as

$$W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}, \quad A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3].$$

Mark the correct statement below.

- The subspace  $W$  can be described as a line through the origin of  $\mathcal{R}^3$ .  
 The dimension of the subspace  $W$  is 3.  
 The linear combination  $\mathbf{v}_1 - \mathbf{v}_2 + 5\mathbf{v}_3$  does not belong to  $W$ .  
  $A$  is an invertible matrix.  
  $\det(A) = 0$ .

Remark: In the problem below there are one or more correct answers in question (a) and (b). Here one false mark will cancel one correct mark.

### Problem 13 (6 points)

A list of vectors is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ 8 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) (3 points, with annulment). Which of the following sets are linearly independent?

- |  |   |
|--|---|
| <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_2\}$                          | <input checked="" type="checkbox"/> $\{\mathbf{v}_1\}$  |
| <input checked="" type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_3\}$               | <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$               |
| <input checked="" type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ | <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ |

(b) (3 points, with annulment). Which of the following sets span  $\mathcal{R}^3$ ?

- |   |  |
|---|--|
| <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_2\}$               | <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$            |
| <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_3\}$               | <input checked="" type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ |
| <input type="checkbox"/> $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ | <input type="checkbox"/> $\{\mathbf{v}_2\}$  |

### Problem 14 (6 points)

In MATLAB's Command Window one types the following:

```
>> t = pi/4;
>> A = [ cos(t) -sin(t) ; sin(t) cos(t) ];
>> v = [1; 1];
>> A*A*v
```

What answer will MATLAB give?

- |  |  |
|--|--|
| <input type="checkbox"/> ans =<br>1.0000<br>2.0000 | <input checked="" type="checkbox"/> ans =<br>-1.0000<br>1.0000 |
| <input type="checkbox"/> ans =<br>1.0000<br>1.0000 | <input type="checkbox"/> ans =<br>1.5000<br>-1.5000            |
| <input type="checkbox"/> ans =<br>0.0000<br>0.0000 | <input type="checkbox"/> ans =<br>2.0000<br>-2.5000            |