For at finde den danske version af prøven, begynd i den modsatte ende!

Please disregard the Danish version on the back if you participate in this English version of the exam.

Exam in Linear Algebra

First Year at The Faculty of IT and Design and at the Faculty of Engineering and Science

June 12, 2017, 9.00-13.00

This test consists of 10 pages and 14 problems. All problems are "multiple choice" problems. Your answers must be given by marking the relevant boxes on these sheets.

It is allowed to use books, notes, xerox copies etc. It is **not allowed** to use any electronic devices such as pocket calculators, mobile phones or computers.

The listed percentages specify by which weight the individual problems influence the total examination.
Your marks concerning the first three problems will be evaluated as follows: Every wrong mark will annul one correct mark.

Remember to fill in your full name (including middle names) together with your student number below. Moreover, please mark the team that you participate in.

NAME: ________________________________

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Problem 1 (6%)

Which of the following statements concerning the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 - x_2 + x_3 &= 0 \\ x_1 + 7x_2 - 4x_3 &= 15 \end{aligned}$$

are correct?

☐ $x_1 = 2, x_2 = -1, x_3 = -5$ solves the system.
☐ $x_1 = 2, x_2 = 2, x_3 = 1$ solves the system
☐ $x_1 = 2, x_2 = -1, x_3 = -5$ is the only solution of the system.

Problem 2 (8%)

This problem concerns an investigation of the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Which of the following statements are correct?

☐ $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \text{Null}(A)$.

☐ $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector for A.

☐ $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ is an eigenvector for A.

☐ $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for the same eigenvalue.

☐ $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.

☐ A is invertible (regular).

☐ A is diagonalizable.

☐ There is a unique diagonal matrix D which is similar to A.

Problem 3 (8%)

A is an $m \times n$-matrix, B is a 2×3 matrix, $C = AB$ is a 3×3 matrix.

1. What are the values of m and n?

   ☐ $m = n = 2$ ☐ $m = 2, n = 3$ ☐ $m = 3, n = 2$ ☐ $m = n = 3$

2. A is the standard matrix of a linear transformation $S : \mathcal{R}^n \to \mathcal{R}^m$, B is the standard matrix of a linear transformation $T : \mathcal{R}^3 \to \mathcal{R}^2$, and C is the standard matrix of a linear transformation $U : \mathcal{R}^3 \to \mathcal{R}^3$.
   Which of the following statements are *wrong*, for sure?

   (a) ☐ S is onto (surjective)
   ☐ S is one-to-one (injective)

   (b) ☐ T is onto (surjective)
   ☐ T is one-to-one (injective)

   (c) ☐ U is onto (surjective)
   ☐ U is one to-one (injective)

Problem 4 (8%)

This problem concerns the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 1 & 0 & 4 \\ 2 & -1 & 1 & 2 \end{bmatrix}$ and the vectors

$\mathbf{b} = \begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$.

1. Is $\mathbf{d}$ contained in the column space Col A?

   ☐ Yes ☐ No

2. Is $\mathbf{b}$ contained in the column space Col A?

   ☐ Yes ☐ No

3. Is $\mathbf{d}$ contained in the null space Null A?

   ☐ Yes ☐ No

4. Is $\mathbf{c}$ contained in the null space Null A?

   ☐ Yes ☐ No

Problem 5 (6%)

The matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is the standard matrix for a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$.

1. What is the value of n?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5

2. What is the value of m?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5

3. What is the rank of A?

   ☐ 2 ☐ 3 ☐ 4 ☐ 5

4. What is the dimension of the null space of A?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3

5. Is T onto (surjective)?

   ☐ Yes ☐ No

6. Is T one-to-one (injective)?

   ☐ Yes ☐ No

Problem 6 (4%)

Which of the following vectors agrees with the product of the matrix $A = \begin{bmatrix} 3 & -1 \\ -5 & 6 \\ 4 & -2 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

☐ $\begin{bmatrix} 6 \\ -1 \end{bmatrix}$ ☐ $\begin{bmatrix} 5 \\ -4 \\ 6 \end{bmatrix}$ ☐ $\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ ☐ none of them

Problem 7 (6%)

This problem is about $m \times m$ matrices A and B, their products $C = BA$ and $D = AB$ and some of their eigenvectors:
$\mathbf{v}$ is an eigenvector for A with corresponding eigenvalue 2 and $\mathbf{w} = A\mathbf{v}$ is an eigenvector for B with corresponding eigenvalue -4. Is it always true that

1. $\mathbf{v}$ is an eigenvector for C?

   ☐ Yes ☐ No

   In the affirmative case, what is the corresponding eigenvalue?

   ☐ -2 ☐ 2 ☐ 8 ☐ -8

2. $\mathbf{v}$ is an eigenvector for D?

   ☐ Yes ☐ No

   In the affirmative case, what is the corresponding eigenvalue?

   ☐ -2 ☐ 2 ☐ 8 ☐ -8

3. $\mathbf{w}$ is an eigenvector for $B^2 = BB$?

   ☐ Yes ☐ No

   In the affirmative case, what is the corresponding eigenvalue?

   ☐ -4 ☐ 4 ☐ 2 ☐ 16

Problem 8 (6 %)

The three vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}$ form a basis $\mathcal{B}$ for $\mathcal{R}^3$.
Applying the Gram-Schmidt process to $\mathcal{B}$, yields an orthogonal basis consisting of vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. Which of the following statements are correct?

1. ☐ $\mathbf{v}_1 = \mathbf{u}_1$ ☐ $\mathbf{v}_1 = -\mathbf{u}_1$ ☐ none is correct

2. ☐ $\mathbf{v}_2 = \mathbf{u}_2$ ☐ $\mathbf{v}_2 = \mathbf{u}_2 - \mathbf{u}_1$ ☐ none is correct

3. ☐ $\mathbf{v}_3 = \mathbf{u}_3$ ☐ $\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{u}_1 - \mathbf{u}_2$
   ☐ $\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{u}_2$ ☐ none is correct

Problem 9 (12 %)

This problem concerns the following four matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

Is it correct that

1. A and B are inverse to each other?

   ☐ Yes ☐ No

2. C and D are inverse to each other?

   ☐ Yes ☐ No

3. A is diagonalizable?

   ☐ Yes ☐ No

4. B is diagonalizable?

   ☐ Yes ☐ No

5. C is diagonalizable?

   ☐ Yes ☐ No

6. D is diagonalizable?

   ☐ Yes ☐ No

Problem 10 (8 %)

This problem deals with vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ in $\mathcal{R}^2$ and with a linear transformation $T : \mathcal{R}^2 \to \mathcal{R}^2$ satisfying:

$$T(\mathbf{u}) = \mathbf{u} \text{ and } T(\mathbf{v}) = -\mathbf{v}.$$

Let $A = [\mathbf{a}_1 \ \mathbf{a}_2]$ denote the standard matrix for the transformation T; hence $\mathbf{a}_1$ and $\mathbf{a}_2$ denote the column vectors of the matrix A.

Which of the following statements are correct?

1. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (orthogonal) to each other.

   ☐ Yes ☐ No

2. $\mathbf{u}$ is an eigenvector for T.

   ☐ Yes ☐ No

3. $\mathbf{u} + \mathbf{v}$ is an eigenvector for T.

   ☐ Yes ☐ No

4. $\mathbf{a}_1 = \begin{bmatrix} 0.96 \\ 0.28 \end{bmatrix}$

   ☐ Yes ☐ No

5. $\mathbf{a}_2 = \begin{bmatrix} 0.96 \\ 0.28 \end{bmatrix}$

   ☐ Yes ☐ No

6. A is an orthogonal matrix.

   ☐ Yes ☐ No

7. T describes a rotation in the plane.

   ☐ Yes ☐ No

8. T describes a reflection in the plane.

   ☐ Yes ☐ No

Problem 11 (10 %)

This problem concerns the matrix $A = \begin{bmatrix} 7 & -7 & 3 & 1 \\ 2 & -2 & 1 & 0 \\ 6 & -6 & 3 & 1 \end{bmatrix}$

and the vector $\mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 7 \end{bmatrix}$.

Reducing the augmented matrix $[A \ \mathbf{b}] = \begin{bmatrix} 7 & -7 & 3 & 1 & 6 \\ 2 & -2 & 1 & 0 & 1 \\ 6 & -6 & 3 & 1 & 7 \end{bmatrix}$

results in the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

in reduced echelon form.

1. Which of the columns of A are pivot columns?

   ☐ all four columns ☐ exactly columns 1, 3 and 4
   ☐ only column 2 ☐ exactly columns 1, 2 and 4

2. What is the rank of the matrix A?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

3. What is the nullity of the matrix A?

   ☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

4. Is there a solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the equation $A\mathbf{x} = \mathbf{b}$ with $x_2 = 2$?

   ☐ Yes ☐ No

   In the affirmative case, which of the following alternatives is correct for the remaining coordinates?

   ☐ $x_1 = 1, x_3 = 3, x_4 = 4$ ☐ $x_1 = 1, x_3 = x_4 = 0$
   ☐ $x_1 = -1, x_3 = 3, x_4 = 4$

Problem 12 (6%)

This problem is an investigation of relationships between the three vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ c \end{bmatrix} \text{ in } \mathcal{R}^3.$$

Observe that the last coordinate c of the vector $\mathbf{c}$ is a reel variable.
Which of the following statements are correct?

1. $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linearly dependent vectors for

   - ☐ $c = 0$
   - ☐ $c = 4$
   - ☐ $c \neq 4$
   - ☐ all real numbers c
   - ☐ no real number c

2. $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ span $\mathcal{R}^3$ for

   - ☐ $c = 0$
   - ☐ $c = 4$
   - ☐ $c \neq 4$
   - ☐ all real numbers c
   - ☐ no real number c

Problem 13 (8%)

This problem concerns (6×6)-matrices A, B and $C = AB$.
It is known that $\det A = -2$ and $\det C = -6$.

1. What is the value of $\det(-A)$?

   ☐ -2 ☐ 2 ☐ -4 ☐ 4

2. What is the value of $\det(2A)$?

   ☐ -4 ☐ 64 ☐ -128 ☐ 128

3. What is the value of $\det B$?

   ☐ 12 ☐ -12 ☐ 3 ☐ -3

4. What is the value of $\det(B^T A)$?

   ☐ -6 ☐ 6 ☐ $\frac{3}{2}$ ☐ $-\frac{3}{2}$

Problem 14 (4 %)

The following commands are entered into MATLAB's Command Window:

```
>> A = [1 2 -1; 3 1 1; 2 -1 1 ];
>> b = [3; 4; 2];
>> T = [A b];
```

1. What is the size of the matrix T?

   ☐ 4×3 ☐ 4×4 ☐ 4×5 ☐ 3×4 ☐ 12×1

2. The system of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$. Which of the following sequence of commands results in a calculation of $\mathbf{x}$?

   ☐ `>> R = rref(T); x=R(:,4)`

   ☐ `>> R = rref(A); x=R(:,4)`

   ☐ `>> R = rref(T); x=R(:,5)`

   ☐ `>> R = rref(A); x=R(4,:)`

   ☐ `>> R = rref(T); x=R(4,:)`